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TABLE DES MATIÈRES

VOLUME VIII

ON CERTAIN DETERMINANTS WHOSE ELEMENTS ARE ORTHOGONAL POLYNOMIALS *

By

S. Karlin and G. Szegö

in Stanford, California

To Paul Turán
on his 50th birthday
August 18, 1960

CONTENTS

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* This research was supported by the National Science Foundation Grants NSF — G 9669 — G 11296.

Chapter 1. PRELIMINARIES

§ 1. Introduction.

The present investigation was inspired by Turán's inequality (see below) and by probability considerations. Our purpose is to study certain determinants whose elements are orthogonal polynomials. In some cases the polynomials represented by these determinants will keep a constant sign, in some other cases they will prove to be oscillatory. In this introductory section we shall try to give a rather detailed account — occasionally at the expense of repetitions — not only of the principal results to be proved but also of the underlying ideas.

1. Let $\{Q_n(x)\}$ be a system of orthogonal polynomials. In all cases $\{Q_n(x)\}$ will denote a polynomial of the exact degree n however its normalization might be different according to prevailing circumstances. A system of this kind is associated with a weight function, more generally with a distribution function; we refer to the latter also as a measure. As usual the set of the points of increase of the distribution function is called the spectrum. This is in some instances an interval (finite, half infinite, or the whole real axis), sometimes a finite set (in which case the system of polynomials is also finite), sometimes a countable set of values which we assume to tend to $+\infty$, and possibly some more complicated set.

We sometimes study general orthogonal polynomials. However, our main attention will be centered about the following special classes:

(a) Ultraspherical polynomials, $P_n^{(\lambda)}(x)$, in particular Legendre polynomials, $P_n(x)$;

(b) Laguerre polynomials, $L_n^{(\alpha)}(x)$;

(c) Hermite polynomials, $H_n(x)$.

We shall refer to the classes (a), (b), (c) as the classical polynomials: the corresponding spectra and the weight functions are well known. Moreover we shall consider:

(d) Orthogonal polynomials associated with a discrete measure.

This is an interesting class arising from a distribution function which is constant in stretches. The set of jump points (spectrum) is a finite or infinite sequence, and in the latter case we assume as mentioned above that the points of this sequence tend to $+\infty$. We point out the following special cases:

(e) Poisson-Charlier polynomials, $c_n(a\,;\,x)$;

(f) Meixner polynomials, $M_n(\beta\,,\gamma\,;\,x)$;

(g) Krawtchouk polynomials, $k_n(N\,,p\,;\,x)$;

(h) Tchebychev's polynomials of a discrete measure, $t_n(x)$.

The systems (e) and (f) are infinite, the systems (g) and (h) finite. The definition and principal properties of these polynomials will be compiled in § 5.

2. Let us consider the system $\{Q_n(x)\}$ of orthogonal polynomials. Various classical theorems describe the behavior of $Q_n(x)$ as a function of x; for instance, certain oscillation properties hold when x runs over the spectrum [13, p. 44].(*) Some other theorems describe $Q_n(x)$ as a function of n; for instance, if n runs over the values 0, 1, 2, ..., a recurrence formula holds which can also be interpreted as a difference equation of the second order in n [13, p. 42]. These and other facts support the evidence that, at least in certain instances of distributions, a duality between the variables x and n prevails. For example, in the classical cases (a), (b), (c) the function $Q_n(x)$ satisfies a linear differential equation of the second order in x to be contrasted with the difference equation in n mentioned above. This duality becomes particularly suggestive for the general class (d) when both x and n range over a discrete set; in the cases (e)—(h) a difference equation of the second order holds both in x and n. The duality becomes perfect in the cases (e), (f) and (g) when, under a proper normalization of $Q_n(x)$ the relation $Q_n(x) = Q_x(n)$ is valid. (This is definitely not true, at least not in this simple form, in the case (h).)

3. Let

$$Q_n(x)\,,\ Q_{n+1}(x)\,,\ \ldots\,,\ Q_{n+l-1}(x) \tag{1.1}$$

be a sequence of orthogonal polynomials associated with a certain measure. The Wronskian of these polynomials is defined in the usual way:

$$W(Q_n(x)\,,Q_{n+1}(x)\,,\ldots\,,Q_{n+l-1}(x)) = \begin{vmatrix} Q_n(x) & Q_{n+1}(x) & \ldots & Q_{n+l-1}(x) \\ Q'_n(x) & Q'_{n+1}(x) & \ldots & Q'_{n+l-1}(x) \\ \cdot & \cdot & \ldots & \cdot \\ Q_n^{(l-1)}(x) & Q_{n+1}^{(l-1)}(x) & \ldots & Q_{n+l-1}^{(l-1)}(x) \end{vmatrix}. \tag{1.2}$$

* Numbers in square brackets refer to the Bibliography at the end of the paper.

The natural analog of this expression for the case (d) will be:

(1.3)
$$Q\begin{pmatrix} n, n+1, \dots, n+l-1 \\ x_k, x_{k+1}, \dots, x_{k+l-1} \end{pmatrix} = \begin{vmatrix} Q_n(x_k) & Q_{n+1}(x_k) & \dots & Q_{n+l-1}(x_k) \\ Q_n(x_{k+1}) & Q_{n+1}(x_{k+1}) & \dots & Q_{n+l-1}(x_{k+1}) \\ \cdot & \cdot & \dots & \cdot \\ Q_n(x_{k+l-1}) & Q_{n+1}(x_{k+l-1}) & \dots & Q_{n+l-1}(x_{k+l-1}) \end{vmatrix}$$

where $x_k, x_{k+1}, \dots, x_{k+l-1}$ are "successive" jump points, i.e. successive points of the spectrum in question. If the spectrum is a continuous one, the successive points coincide and as a limiting case W arises. We refer to (1.3) also as a Wronskian or sometimes as the "discrete Wronskian". The normalization of the polynomials has only a trivial influence on the Wronskian.

For the purpose of particular emphasis we introduce a similar symbol as (1.3) for the transposed determinant:

(1.3′)
$$Q\begin{pmatrix} x_k, x_{k+1}, \dots, x_{k+l-1} \\ n, n+1, \dots, n+l-1 \end{pmatrix} = \begin{vmatrix} Q_n(x_k) & Q_n(x_{k+1}) & \dots & Q_n(x_{k+l-1}) \\ Q_{n+1}(x_k) & Q_{n+1}(x_{k+1}) & \dots & Q_{n+1}(x_{k+l-1}) \\ \cdot & \cdot & \dots & \cdot \\ Q_{n+l-1}(x_k) & Q_{n+l-1}(x_{k+1}) & \dots & Q_{n+l-1}(x_{k+l-1}) \end{vmatrix}.$$

In agreement with the duality mentioned above we may form now a determinant whose successive rows arise from (1.1) by replacing the variable n (instead of x) by successive values. This yields the Turánian:

(1.4)
$$T(Q_n(x), Q_{n+1}(x), \dots, Q_{n+l-1}(x)) = \begin{vmatrix} Q_n(x) & Q_{n+1}(x) & \dots & Q_{n+l-1}(x) \\ Q_{n+1}(x) & Q_{n+2}(x) & \dots & Q_{n+l}(x) \\ \cdot & \cdot & \dots & \cdot \\ Q_{n+l-1}(x) & Q_{n+l}(x) & \dots & Q_{n+2l-2}(x) \end{vmatrix}.$$

This is a determinant of the Hankel type which is in a sense dual to the Wronskian (1.2). The normalization of the polynomials in question is here of great importance.

We shall be concerned with the sign of certain determinants of the form (1.2), (1.3) and (1.4). In frequent instances they keep a constant sign for even l and they are oscillating for odd l. In the latter cases we prove frequently the Sturm character of the set arising for $n = 0, 1, 2, \dots$. (Concerning Sturm sets, see § 2.) Generally speaking, the discussion of the determinants of the Wronski type will be easier than that of the determinants of the Turán type. We took this fact into account in arranging our results

(see Contents). We shall study the Wronski type for special as well as rather general classes of measures whereas the Turán type almost exclusively only for the special classes (a)—(c) and (e)—(h). Sometimes we establish peculiar relations between determinants of the T and W type (§§ 12, 14, 16).

We note that (1.3) is a special case of the determinant of more general type:

$$(1.5) \qquad Q\begin{pmatrix} n_1, n_2, \ldots, n_l \\ x_1, x_2, \ldots, x_l \end{pmatrix} = \begin{vmatrix} Q_{n_1}(x_1) & Q_{n_2}(x_1) & \ldots & Q_{n_l}(x_1) \\ Q_{n_1}(x_2) & Q_{n_2}(x_2) & \ldots & Q_{n_l}(x_2) \\ \cdot & \cdot & \ldots & \cdot \\ Q_{n_1}(x_l) & Q_{n_2}(x_l) & \ldots & Q_{n_l}(x_l) \end{vmatrix},$$

$n_1 < n_2 < \ldots < n_l$; $x_1 < x_2 < \ldots < x_l$, studied in various respects by Karlin-McGregor [8].

4. Our first objective is to prove the following two theorems on Wronskians; the measure involved here is a general one.

Theorem 1: *Let $\{Q_n(x)\}$ denote the system of orthogonal polynomials associated with an arbitrary distribution function $\alpha(x)$ having an infinity of points of increase. Let $Q_n(x) = k_n(-x)^n + \ldots, k_n > 0$. We assume that l is even. Then the Wronskian*

$$(1.6) \qquad W(n, l; x) = W(Q_n(x), Q_{n+1}(x), \ldots, Q_{n+l-1}(x))$$

keeps a constant sign for all real x; more specifically,

$$(1.7) \qquad (-1)^{l/2} W(n, l; x) > 0.$$

The sign $(-1)^{l/2}$ in the present case as well as in other cases, arises from $(-1)^{l(l-1)/2}$ by specializing l to be even; in reversing the order of the columns of W the resulting determinant would be always positive.

Theorem 2: *We make the same assumptions as in Theorem 1 except that l is odd. The polynomials*

$$(1.8) \qquad W(n, l; x), \quad n = 0, 1, 2, \ldots,$$

form a Sturm set on the real axis so that $W(n, l; x)$ has exactly n simple zeros and the zeros of two successive Wronskians $W(n, l; x)$ and $W(n+1, l; x)$ strictly interlace.

The proof of Theorem 2 makes use of Theorem 1.

5. Further we prove two similar theorems for discrete measures.

Theorem 3: *Let $\alpha(x)$ be a discrete measure whose spectrum coincides with a countable set of points $\{a_0 < a_1 < a_2 < \ldots\}$. We denote by $Q_n(x) = k_n(-x)^n + \ldots, k_n > 0$, the associated system of orthogonal polynomials. Let l be even. Then*

$$(1.9) \qquad (-1)^{l/2} Q \begin{pmatrix} n, & n+1, & \ldots, & n+l-1 \\ x_0, & x_1, & \ldots, & x_{l-1} \end{pmatrix} > 0$$

provided that the x_i form a non-decreasing sequence of "successive" values a_r with equalities allowed.

The symbol Q is used as in (1.3). The term "successive" is meant in the following sense: First we take some a_r a certain number of times, then a_{r+1} a certain number of times, etc. The usual interpretation of the determinant applies when two or more successive x's are equal to the same a_r. We cite the following example:

$$(1.10) \qquad (-1)^{l/2} Q \begin{pmatrix} n, & n+1, & \ldots, & n+l-1 \\ a_r, & a_r, & \ldots, & a_r \end{pmatrix} > 0, \qquad l \text{ even}; \; r = 0, 1, 2, \ldots.$$

This determinant is of course identical with the Wronskian (1.2), $x = a_r$; the inequality (1.10) follows from Theorem 1.

Before naming another example we introduce the following notation to be used whether l is even or odd:

$$(1.11) \qquad Q \begin{pmatrix} n, & n+1, & \ldots, & n+l-1 \\ a_r, & a_{r+1}, & \ldots, & a_{r+l-1} \end{pmatrix} = u(n, l; r) = u_n(r).$$

(Sometimes we suppress the letter l.) Now another special case of (1.9) is the following:

$$(1.12) \qquad (-1)^{l/2} Q \begin{pmatrix} n, & n+1, & \ldots, & n+l-1 \\ a_r, & a_{r+1}, & \ldots, & a_{r+l-1} \end{pmatrix} = (-1)^{l/2} u(n, l; r) = (-1)^{l/2} u_n(r) > 0, \qquad l \text{ even}; \; r = 0, 1, 2, \ldots.$$

The following Theorem 4 is related to Theorem 3 in a similar fashion as Theorem 2 is to Theorem 1. We have

Theorem 4: Let $\alpha(x)$, $Q_n(x)$, $u(n, l; r) = u_n(r)$ have the previous meaning, and let l be odd. The sequences

$$\{u_n(r)\,;\, r = 0, 1, 2, \ldots\}, \qquad n = 0, 1, 2, \ldots, \tag{1.13}$$

form a Sturm set in the following sense. The function $u_n(x)$ defined for $x \geqq a_0$ by linear interpolation has exactly n nodal zeros[1] and the zeros of $u_n(x)$ and $u_{n+1}(x)$ strictly interlace.

Applying Theorem 3 to the special case of the Poisson-Charlier polynomials, we obtain immediately, in view of a well known relation of these polynomials to those of Laguerre [cf. (5.21)], a special case of Theorem 5, namely that of the Laguerre polynomials $L_n^{(\alpha)}(x)$ with integer α. This result will be of some importance at a later occasion when we shall prove Theorem 5 for general Laguerre polynomials.

6. Let us consider now determinants of the type (1.4). The simplest non-trivial case occurs in Turán's inequality, $P_n(x)$ is Legendre's polynomial,

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} \leqq 0, \quad n \geqq 0, \quad -1 \leqq x \leqq 1, \tag{1.14}$$

where the equality sign holds only for $x = \pm 1$. The literature on this topic is considerable; we refer here to [14] where four different proofs of this inequality have been offered.

The main part of Chapter 2 is devoted to the proof of the following two theorems. The first is a generalization of Turán's inequality, the second is a generalization of the well known oscillation property of the orthogonal polynomials. In these theorems we consider the classical polynomials (a), (b), (c) defined above.

Theorem 5: Let $\{Q_n(x)\}$ denote one of the following special classes of orthogonal polynomials:

$Q_n(x) = P_n^{(\lambda)}(x) / P_n^{(\lambda)}(1)$ in case (a), $\lambda > -\frac{1}{2}$,

in particular $(\lambda = \frac{1}{2})\, Q_n(x) = P_n(x)$;

$Q_n(x) = L_n^{(\alpha)}(x) / L_n^{(\alpha)}(0)$ in case (b), $\alpha > -1$;

$Q_n(x) = H_n(x)$ in case (c).

Let l be an even number. Then the determinants (1.4) of

---

1. For the concepts of linear interpolation and nodal zeros, cf. § 2.

the Turán type, for which we shall write now $T(n, l; x)$, are of the constant sign $(-1)^{l/2}$ provided $-1<x<1$ in case (a), $x>0$ in case (b), x arbitrary real in case (c).[2]

As we see, in each particular case x is restricted to the spectral interval.

Theorem 6: We use the symbols $Q_n(x)$ and $T(n, l; x)$ in the same sense as in Theorem 5; let l be odd. The polynomials $\{T(n, l; x);\ n=0, 1, 2, \ldots\}$ form a Sturm set in the following sense: $T(n, l; x)$ has exactly n simple zeros in the interval $-1<x<1$, $x>0$, x real, respectively, and the zeros of $T(n, l; x)$ and $T(n+1, l; x)$ strictly interlace.

7. Let us indicate in a brief digression some ideas leading to Theorem 5. We consider first the case $l=2$. The third proof given for Turán's inequality in [14] was based on the following suggestion of Prof. G. Pólya. Let $f(z)$ be an entire function of the form

$$(1.15) \qquad f(z) = \sum_{n=0}^{\infty} \frac{u_n}{n!} z^n = e^{-\alpha z^2 + \beta z} \prod (1 + \beta_k z) e^{-\beta_k z}$$

where $\alpha \geqq 0$, β and β_k real, $\Sigma \beta_k^2 < \infty$. For the coefficients u_n we have the inequality $u_n^2 - u_{n-1} u_{n+1} \geqq 0$. This class has been considered by Laguerre, Pólya and I. Schur [10]; the functions $cz^r f(z)$ are the only entire functions which are limits of polynomials with only real zeros. It can be shown that the associated (Jensen-) polynomials

$$(1.16) \quad f_N(z) = u_0 + \binom{N}{1} u_1 z + \binom{N}{2} u_2 z^2 + \ldots + \binom{N}{N-1} u_{N-1} z^{N-1} + u_N z^N$$

have only real zeros.

The fact that (1.16) has only real zeros, yields in itself immediately the inequalities $u_n^2 - u_{n-1} u_{n+1} \geqq 0$, $1 \leqq n \leqq N-1$; we have to use only Rolle's theorem combined with the trivial fact that the inequality holds for the polynomial $u_{n-1} + 2u_n z + u_{n+1} z^2$. The usual argument [cf. 11, vol. 2, Chapter V, Problem 61, p. 232] shows also that the sign $>$ holds unless all zeros of (1.16) coincide.

2. The only exception is the case (a) for $l \geqq 4$, l even (Tchebychev's polynomials, cf. the footnote in § 5). Indeed, from T we can factor out λ^{l-2} provided $l \geqq 2$, l being even or odd.

If we can show that $f(z)$ is of the Laguerre-Pólya-I. Schur class, all inequalities follow immediately, at least in the weaker form ($\geq$ instead of $>$). If $f(z)$ has at least two distinct simple zeros, the stronger inequality holds; indeed, $\lim_{N\to\infty} f_N\left(\frac{z}{N}\right) = f(z)$ uniformly in every circle $|z| \leq R$, so that, in view of the well known theorem of Hurwitz we conclude that $f_N(z)$ must have at least two distinct zeros provided N is sufficiently large.

In some cases we can prove directly that all zeros of (1.16) are real and not all coinciding; there is no need then for a reference to $f(z)$ at all and the proof of the inequality (in the sharp form) can be performed in a purely algebraic way.

As an example we consider (1.14). The function

$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} z^n = e^{xz} J_0[(1-x^2)^{1/2} z] \tag{1.17}$$

(J_0 is Bessel's function; see [13, (4.10.7)]) is of the form (1.15) and for $-1<x<1$ it has infinitely many zeros which are all real and simple. Thus Turán's inequality follows in the sharper form. The same conclusion can be drawn from the elementary identity [3]

$$\sum_{n=0}^{N} \binom{N}{n} P_n(x) z^n = (1+2xz+z^2)^{N/2} P_N\left(\frac{1+xz}{(1+2xz+z^2)^{1/2}}\right); \tag{1.18}$$

this is essentially the fourth proof given in [14]. Actually the trivial inequality

$$\left(\frac{\alpha_1+\alpha_2+\ldots+\alpha_N}{\binom{N}{1}}\right)^2 > \frac{\alpha_1\alpha_2+\alpha_1\alpha_3+\ldots}{\binom{N}{2}}$$

valid for the real numbers α_r which are not all the same, suffices in this case.

8. Let l be arbitrary even. We consider again the polynomials (1.16) with only real zeros. One might expect then that the determinants

$$\begin{vmatrix} u_n & u_{n+1} & \ldots & u_{n+l-1} \\ u_{n+1} & u_{n+2} & \ldots & u_{n+l} \\ \cdot & \cdot & \ldots & \cdot \\ u_{n+l-1} & u_{n+l} & \ldots & u_{n+2l-2} \end{vmatrix}, \quad n+2l-2 \leq N,$$

3. See [14]; this identity occurs also in a recent paper of W. N. Bailey [1; cf. (3.3) on p. 238].

will be all of the sign $(-1)^{l/2}$. This would have Theorem 5 as a consequence. However a simple counter-example shows that this is generally not true so that "direct" methods must be employed for the proof of Theorem 5.

These methods will be based on some identities, different in each case (a), (b), (c) connecting the determinant $T(n, l; x)$ of the Turán type with an appropriate determinant of the Wronski type. This is in line with the duality stressed above. Now it will be necessary to show that a certain Wronskian keeps a constant sign on an appropriate range; this will be done by a continuity reasoning (probably a not too desirable argument in this type of problems). There are cases when the continuity reasoning can be avoided: for example, Legendre polynomials and Laguerre polynomials $L_n^{(\alpha)}(x)$ with integer α (see above).

The case (c) of the Hermite polynomials is a limiting case of (a) since (5.34) holds. This remark yields the inequality in question in the weaker form. The method leading to the sharper form of the inequality will be based on a simple matrix multiplication.

The proof of Theorem 6 follows a rather unified pattern.

One more remark about the special case $l = 2$ of Theorem 5, i.e. Turán's inequality $[Q_n(x)]^2 - Q_{n-1}(x)\,Q_{n+1}(x) > 0$. If x is real and not on the spectrum, this inequality is reversed for the ultraspherical polynomials ($\lambda > 0$), and it remains valid for the Laguerre polynomials. The first fact is due to [3] and follows immediately from (5.8) by Schwarz's inequality; the second fact follows by the same argument which was used for $x > 0$ in [14].

9. The following theorem deals with four cases of discrere measures. The orthogonal polynomials in question are defined in § 5.3.

Theorem 7: Let $\{Q_n(x)\}$ denote the system of orthogonal polynomials of

(a) Poisson-Charlier, (b) Meixner, (c) Krawtchouk, (d) Tchebychev, respectively. The polynomials are normalized according to the condition $Q_n(0) = 1$. We have the following inequality of the Turán type, $l = 2$,

$$T(Q_n(x), Q_{n+1}(x)) = \begin{vmatrix} Q_n(x) & Q_{n+1}(x) \\ Q_{n+1}(x) & Q_{n+2}(x) \end{vmatrix} < 0$$

where x is an integer, $x \geqq 1$; moreover $x \leqq N-1$, $n \leqq N-2$ in the case (c) and $x \leqq N-2$, $n \leqq N-3$ in the case (d).

The following stronger assertion applies only to the cases (a), (b), and (c).

Theorem 8: We use the same notation as in Theorem 7, and let l be an even integer. In the cases (a), (b), (c) we have the inequality

$$(-1)^{l/2}\, T\,(Q_n(x)\,,\, Q_{n+1}(x)\,,\,\dots\,,\, Q_{n+l-1}(x)) > 0$$

where x is an integer, $x \geqq l-1$ and in addition $x \leqq N-l+1$, $n \leqq N-2l+2$ in the case (c).

10. We continue with another generalization of the determinants of the Wronski and Turán type discussed so far, restricting ourselves first to $l=2$. We consider the "augmented Wronskian" of the form

$$(1.19) \qquad \varphi_n(k\,;x) = \begin{vmatrix} Q_k(x) & Q_n(x) \\ Q'_k(x) & Q'_n(x) \end{vmatrix}, \quad n = k+1\,,\, k+2\,,\,\dots,$$

and the "augmented Turánian" of the form

$$(1.20) \qquad \psi_n(k\,;x) = \begin{vmatrix} Q_k(x) & Q_n(x) \\ Q_{k+1}(x) & Q_{n+1}(x) \end{vmatrix}, \quad n = k+1\,,\, k+2\,,\,\dots.$$

These determinants have a probabilistic background to which we may return at another occasion.

Theorem 9: Let $\{Q_n(x)\}$ be one of the three classical systems of polynomials defined in Theorem 5. We consider the sequence (1.19) of the augmented determinants $\varphi_n(k\,;x)$ of the Wronski type. They form a weak Sturm set (§2.1) in the intervals $-1<x<1$, $x>0$, x real, respectively; thus $\varphi_n(k\,;x)$ has exactly $n-k-1$ nodal zeros in the respective interval and the zeros of $\varphi_n(k\,;x)$ and $\varphi_{n+1}(k\,;x)$ strictly interlace. Moreover $\varphi_n(k\,;x)$ has no zeros exterior to the spectral interval.

Theorem 10: Under the assumptions of the foregoing theorem, the sequence (1.20) of the augmented determinants $\psi_n(k\,;x)$ of the Turán type forms a Sturm set in the respective interval; thus $\psi_n(k\,;x)$ has exactly $n-k-1$

simple zeros and the zeros of $\psi_n(k;x)$ and $\psi_{n+1}(k;x)$ strictly interlace. Moreover $\psi_n(k;x)$ has no zeros exterior to the spectral interval.

11. Theorem 9 has an analog (dual) for polynomials of a discrete measure which becomes particularly suggestive if we introduce the difference operator $\Delta Q_n = Q_{n+1} - Q_n$. We prove

Theorem 11: Let $\{Q_n(x)\}$ be any system of orthogonal polynomials associated with a discrete measure having the jump points $a_0 = 0 < a_1 < a_2 < \ldots$. We normalize these polynomials by the condition $Q_n(0) = 1$ and consider the following determinants of order 2:

$$(1.21)\quad \Phi_r(k;n) = \Phi_r(n) = \begin{vmatrix} Q_n(a_k) & Q_n(a_r) \\ Q_{n+1}(a_k) & Q_{n+1}(a_r) \end{vmatrix} = \begin{vmatrix} Q_n(a_k) & Q_n(a_r) \\ \Delta Q_n(a_k) & \Delta Q_n(a_r) \end{vmatrix},$$

where k is fixed, $r \geqq k+1$. Then the sequences

$$\{\Phi_r(n);\, n = 0, 1, 2, \ldots\}, \qquad r = k+1,\ k+2, \ldots,$$

form a weak discrete Sturm set (§ 2).

The jump points a_k and a_r correspond now to the degrees k and n appearing in Theorem 9. On the other hand n takes over now the role of x, the operator Δ the role of differentiation. This is in agreement with the duality defined in § 1.

The following consequence can be pointed out. Let $\Phi_r(y)$ be the function arising from the sequence $\Phi^r(n)$ by the usual linear interpolation (§ 2.2)

$$(1.22)\quad \Phi_r(y) = \rho\Phi_r(n) + (1-\rho)\,\Phi_r(n+1) \quad \text{where} \quad y = \rho n + (1-\rho)(n+1), \quad 0 \leqq \rho \leqq 1.$$

Then the function $\Phi_r(y)$ will have exactly $r-k-1$ nodal zeros (some of which might be nodal intervals); thus $\Phi_r(y)$ changes its sign exactly $r-k-1$ times.

Further, the following generalization of Theorem 9 holds:

Theorem 12: Let $\{Q_n(x)\}$ be one of the three classical systems of polynomials considered in Theorem 5. We form the following determinant of the Wronski type of even order $l+2$:

(1.23) $\varphi_n(k, l; x) = \varphi_n(x) = W(Q_k(x), Q_n(x), Q_{n+1}(x), \ldots, Q_{n+l}(x)),$
$$n = k+1, \; k+2, \ldots .$$

These polynomials constitute a weak Sturm set when x belongs to the respective spectrum. Hence $\varphi_n(x)$ has exactly $n-k-1$ nodal zeros on that spectrum.

In the proof of Theorem 12 we make use of the differential equations satisfied by the classical polynomials. This theorem has an analog valid for any system of orthogonal polynomials associated with a discrete measure. This is Theorem 13 in the proof of which the standard recursion formula (difference equation) of the orthogonal polynomials will be employed.

Theorem 13: Let $\{Q_k(x)\}$ be any system of orthogonal polynomials associated with a discrete measure having the jump points $a_0 < a_1 < a_2 < \ldots$. We form the following determinants of the Wronski type of even order $l+2$:

(1.24) $$\psi_n(k, l; r) = \psi_n(r) = Q\begin{pmatrix} a_k, & a_n, & a_{n+1}, \ldots, a_{n+l} \\ r, & r+1, & r+2, \ldots, r+l+1 \end{pmatrix},$$
$$n = k+1, \; k+2, \ldots .$$

Then the sequences $\{\psi_n(r); r = 0, 1, 2, \ldots\}$, $n = k+1$, $k+2, \ldots$, as functions of n form a weak discrete Sturm set.

We use here the notation (1.3′) which underlines the duality to Theorem 12. The usual consequences concerning the existence and number of nodal zeros hold, cf. § 2.

Theorem 13, of course, contains Theorem 11. The proof of Theorem 13 is based on Theorem 11.

12. In the remaining part of Chapter 4 we consider two more types of augmented determinants.

Theorem 14: Let $\{Q_n(x)\}$ be one of the three classical systems of polynomials defined in Theorem 5. Let l and r be fixed integers, $r \geq l-1$. The determinant of Turán type:

(1.25) $$T(Q_0(x), Q_1(x), \ldots, Q_{l-2}(x), Q_r(x))$$

has exactly $r-l+1$ simple zeros on the spectrum.

Theorem 15: Let $\{Q_n(x)\}$ be a system of orthogonal polynomials associated with a discrete measure having

the jump points $a_0 < a_1 < a_2 < \dots$. Let l and r be fixed integers $r \geq l-1$. The sequence of the determinants of the discrete Wronski type

$$(1.26)\quad \psi(r, l; n) = \psi_r(n) = Q\begin{pmatrix} a_0, & a_1, & \dots, & a_{l-2}, & a_r \\ n, & n+1, & \dots, & n+l-2, & n+l-1 \end{pmatrix}, \qquad n = 0, 1, 2, \dots,$$

has exactly $r-l+1$ sign changes.

In contrast to the previous results on augmented determinants Theorems 14 and 15 hold for arbitrary l, even or odd.

In the first case, for all the three systems (a), (b), (c), the determinant (1.25) will be calculated in explicit terms. The special case of Legendre's polynomials with $r = l-1$ will be studied, as an illustration, in § 4.

In the second case we follow the notation (1.3′) so that the duality of the two determinants becomes apparent. A similar duality prevails between Theorems 12 and 13.

Both cases (1.25) and (1.26) can be regarded as augmented determinants of the "principal (initial)" type inasmuch as they begin with $Q_0(x)$ and $Q_n(a_0)$, respectively.

13. Instead of constructing the determinants based on the sequence (1.1) which involves successive functions in the sense of n, we may consider the sequence

$$(1.27)\qquad Q_n(x),\ Q'_n(x),\ \dots,\ Q_n^{(l-1)}(x)$$

which are successive in the sense of x. Forming the following rows by taking successive n values or x values, we obtain again a Wronskian or else a determinant of the Hankel type, respectively. The oscillation properties of this Hankel type determinant are also studied in some cases in the first part of § 30. Our results in this direction are not complete.

As indicated already, a striking contrast between the theorems involving Wronskian determinants (Chapter 2) and those of Turán type (Chapter 3) is the extent of their applicability. The Wronskian theorems hold usually under quite general circumstances of orthogonal polynomial systems. Thus it is suggestive that these results probably can be extended to the situation where the polynomials are replaced by eigenfunctions of general Sturm-Liouville systems. Under certain slight restrictions this is

indeed so and at another opportunity we shall return to this point of view. On the other hand, the theorems of Turán type appear to be firmly bound to the classical orthogonal polynomials, to their discrete analog and to certain systems which can be derived from them by suitable limiting operations. We find in this last category the Bessel functions which may be obtained as limits of Laguerre polynomials. We discuss this fact and its consequences briefly in the remarks of § 30.3.

In addition, another limit relation, connecting the Meixner polynomials and the Laguerre polynomials, is exploited. Specifically, a weak version of the higher order Turán inequality for the Laguerre polynomials (allowing the sign =) is obtained by a suitable limiting procedure appealing to the Turán inequality for the Meixner polynomial system.

In this same section (§ 30.4), we offer some comments concerning alternative methods of normalizations which may be imposed on a polynomial system that already satisfies the Turán inequality ($l = 2$) thus producing a new system of polynomials which also satisfy the same inequality. Applications are then made to the classical polynomials.

It is shown also (§ 30.5) that the Turán inequality fails in the case of the Poisson-Charlier polynomials for any non integer positive value of x.

In the first Appendix (§ 31) a finer study is made pertaining to the oscillation properties of a general system $\{Q_n(x)\}$ of orthogonal polynomials both with respect to the variable x for each fixed n and with respect to the variable n for each fixed x. If $\{Q_n(x)\}$ belongs to a measure with a discrete spectrum, the interplay in the variables n and x is most manifest. The results are developed in all circumstances whether or not x belongs to the spectrum. We refer the reader directly to this section without entering here into a detailed description of the results.

The second Appendix (§ 32) deals with determinants of the Turán type when x is real and outside of the spectrum. A theorem of the convolution type plays here a central role. One of the results has been mentioned already above (end of **8**).

In the third Appendix (§ 33) we present a few open problems which arose in the course of our investigations.

14. After this rather lengthy survey of the content of the present paper (which by no means covers the results of the text completely) we shall indicate briefly the organization of the material. In § 2 we deal with

preliminaries on Sturm sets of functions as well as of sequences. In § 3 certain counter-examples are discussed. In § 4 we deal with some elementary remarks on matrices which will be useful later. Then § 5 contains a list of definitions and elementary properties of orthogonal polynomials to be considered in the course of this paper.

Chapter 2 deals with determinants of the Wronski type. First we prove some preparations on Wronskians in § 6 ; § 7 contains the proof of two theorems associated with a general measure, §§ 8,9 the proofs of two others associated with a discrete measure. In § 10 a new induction process is defined based on a theorem of Christoffel [13, pp. 29—31]. As an application another proof of Theorem 3 is given, moreover a generalization of the same theorem. In a closing section (§ 11) the latter results are applied to the Poisson-Charlier polynomials and this way we succeed in proving a special case of Theorem 5, namely that referring to Laguerre polynomials $L_n^{(\alpha)}(x)$ with an integer α.

The principal subject of Chapter 3 is the proof of Theorem 5 dealing with determinants of the Turán type. In the cases (a) and (b) the proof is rather involved ; it is simple in the case of the Hermite polynomials (§ 18). First, we study Legendre polynomials and prove the theorem in this case (§§ 12,13). This will be used in dealing with the general case of ultraspherical polynomials (§§ 14,15), just as much as the case of the Laguerre polynomials with integer α is used as the foundation for the general case of Laguerre polynomials (§§ 16,17). In both cases (a), (b) the reasoning is based on a certain identity connecting the given determinant of the Turán type with another one of the Wronski type. This identity holds whether l is even or odd. The identities mentioned are proved in §§ 12, 14, 16, the non-vanishing of the determinants in question in §§ 13, 15, 17. In § 19 we deal with the case of l odd when a Sturm set arises.

Two further sections of Chapter 3 deal with discrete measures, first for the four special cases mentioned in Theorem 7 (§ 20) the order l being $= 2$, and then for the three first cases with an arbitrary even l.

Chapter 4 deals with determinants of the "augmented" type ; here the duality mentioned several times before appears to a rather remarkable extent. We spare the formulation of the content of these sections ; the corresponding theorems are given above.

Four closing sections follow. The first (§ 30) is devoted to various

disconnected remarks. The second (§ 31) deals again with the duality stressed before.

The bibliography is by no means complete and it contains only certain items to which explicit reference is made in the text.

One word about the notation. Any matrix occuring in the sequel will be denoted by $(a_{\mu\nu})$, the corresponding determinant by $[a_{\mu\nu}]$; the running letters μ, ν indicating the rows and columns are replaced sometime by α, β or by p, q. Most occuring matrices will be of the square type.

§ 2. Sturm sets.

The following definitions differ from the usual ones more in emphasis rather than in substance. First, we consider sets of functions and then sets of their discrete analogs, namely sequences.

1. Definition: An ordered set of real valued functions $\{f_n(x);\ n=0, 1, 2, \ldots\}$ each defined and analytic in the open interval $I=(a,b)$ is said to form a Sturm set in I if the following conditions are satisfied:

(a) $f_n(x)$ has precisely n simple zeros located in I;

(b) the zeros of $f_n(x)$ and $f_{n+1}(x)$ in I strictly interlace.

We exclude the function identically zero. In some cases the set is finite. It may occur that the functions in question vanish at the end points a and b; we shall encounter such a situation in Theorem 6 (§ 19) and in Theorem 10 (§ 24). The Sturm sets we shall analyze always have the feature that none of their member functions vanish exterior to I. For definiteness we assume in this section that a and b are finite; the changes if $a=-\infty$ or $b=+\infty$ are only slight.

Here are a few hints about the method we shall employ (sometimes with unessential modifications) in verifying the Sturm character of a given set of functions. We first establish directly that $f_0(x)$ never vanishes in I. Next we demonstrate that x_0 being a zero of $f_n(x)$ in I, we have

$$f_{n-1}(x_0)\, f_{n+1}(x_0) < 0 \tag{2.1}$$

so that no two consecutive members can vanish for the same x_0 in I. Further we show that x_0 being a zero of $f_n(x)$ in I,

$$f_{n-1}(x_0)\, f'_n(x_0) \tag{2.2}$$

does not vanish and has a sign independent of the selection of the zero x_0. (Hence x_0 is a simple zero.) Similarly we show (or we conclude from the previous results) that

$$f_{n+1}(x_0)\, f'_n(x_0) \tag{2.2'}$$

has a single sign independent again of the selection of x_0.

The further procedure will be inductive. From the assertion on (2.2′) we find that between two successive zeros of $f_n(x)$ we have at least one zero of $f_{n+1}(x)$. On the other hand the assertion regarding (2.2), with n replaced by $n+1$, implies that between two successive zeros of $f_{n+1}(x)$ there lies at least one zero of $f_n(x)$. These two facts show that the zeros of $f_n(x)$ and $f_{n+1}(x)$ strictly interlace. It remains to establish the existence of the proper number of zeros. This can be done in various ways, for instance by verifing that all $f_n(x)$ keep a constant sign in the left neighborhood of b and this sign is either constant in n or it alternates with n. Indeed, let ξ_n be the largest zero of $f_n(x)$, let $\xi_{n-1} < \xi_n$ and let us assume that all $f_n(x)$ are positive near $x = b$. Then for $x = \xi_n$, the function $f_{n-1}(x)$ will be positive, thus $f_{n+1}(x)$ negative so that there must be at least one zero of $f_{n+1}(x)$ (hence exactly one) between ξ_n and b, i.e. $\xi_{n+1} > \xi_n$. If the signs near b alternate we apply the previous remark to $(-1)^n f_n(x)$.

We shall distinguish between two kinds of zeros. A zero x_0 of a function $f(x)$ is called n o d a l if $f(x)$ has opposite signs for $x = x_0 + h$ and $x = x_0 - h$, h sufficiently small; in the contrary event, x_0 is called n o n - n o d a l. Obviously x_0 is a nodal zero if and only if it has an odd multiplicity. A set $\{f_n(x);\ n = 0, 1, 2, \ldots\}$ is called a w e a k S t u r m s e t in a certain open interval I if condition (a) is weakened to the extent that in its statement "simple zeros" is replaced by "nodal zeros"; (b) remains unchanged except that zeros now mean nodal zeros.

2. Discrete Sturm sets. We consider the discrete analog of a Sturm set replacing a function $f(x)$ by a sequence

$$u_0,\ u_1,\ u_2,\ \ldots,\ u_r,\ \ldots \tag{2.3}$$

whose elements are real numbers. We say as usual [cf. 11, Vol. 2, Chapter V, p. 37] that at r a sign variation in (2.3) takes place if either

$$u_{r-1}\, u_r < 0 \tag{2.4}$$

or $k \geqq 2$ exists such that

$$(2.4') \qquad u_{r-k+1} = u_{r-k+2} = \dots = u_{r-1} = 0\,, \quad u_{r-k}\,u_r < 0\,.$$

This definition would be completely satisfactory for our purposes. However, for the sake of analogy and also for easier formulation we introduce a function $u(x)$, $x \geqq 0$, arising from (2.3) by "linear interpolation" as follows. We write

$$(2.5) \quad u(x) = \rho u_{r-1} + (1-\rho)\,u_r \text{ provided } x = \rho(r-1) + (1-\rho)r,\ 0 \leqq \rho \leqq 1,$$

i.e. we join the points $x = r$, $u = u_r$ representing the sequence, by straight lines. This function is linear in stretches. In the second case (2.4′) we call $r - k + 1 \leqq x \leqq r - 1$ a nodal interval; this interval shrinks to a point in the open interval $(r - 1\,, r)$ if (2.4) holds. The nodal intervals of $u(x)$ are disjoint in the strict sense.

We say that an ordered set of sequences

$$(2.6) \qquad \{u_n(r)\,;\ r = 0,\,1,\,2,\,\dots\}\,, \qquad n = 0,\,1,\,2,\,\dots\,,$$

form a discrete Sturm set if the set of the functions $u_n(x)$ arising from (2.6) by linear interpolation satisfies the following conditions:

(a) $u_n(x)$ has precisely n nodal intervals located in $x > 0$;

(b) the nodal intervals of $u_n(x)$ and $u_{n+1}(x)$ strictly interlace.

The verification of these two properties requires mostly a similar argument as in the continuous case. The set (2.6) is called a weak discrete Sturm set if not all nodal intervals of all $u_n(x)$ reduce to points [i.e. if for some n and r, condition (2.4′) holds, $k \geqq 2$].

In certain special situations we may interpolate by appropriate analytic (non linear) functions and again assert the Sturm character of the resulting functions. This change of the interpolation modifies of course the location of the nodal zeros but in no way the essential content of the assertions (a) and (b).

§ 3. Counter-examples.

In this section we discuss three simple counter-examples.

1. Polynomials with only real zeros. It is enlightening to point out the following negative fact: Let the polynomial (1.16) have only real zeros. The inequality of Turán type

$$\begin{vmatrix} u_n & u_{n+1} \\ u_{n+1} & u_{n+2} \end{vmatrix} < 0$$

can not be extended to higher (even) order determinants. Thus we have to use some special methods in proving inequalities of the type of Theorem 5.

The required counter-example is furnished by considering

$$f(z)=\frac{1}{2}[(z+i)^N+(z-i)^N]=u_0+\binom{N}{1}u_1 z+\binom{N}{2}u_2 z^2+\ldots+u_N z^N.$$

This polynomial is of the exact degree N and has only real zeros since $|z+i|=|z-i|$ can hold only for real z. The zeros are all simple. We have $u_{N-\nu}=0$ for ν odd, and $u_{N-\nu}=i^\nu=(-1)^{\nu/2}$ for ν even. Let, for instance, $N=6$. Clearly the determinant

$$\begin{vmatrix} u_0 & u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 & u_4 \\ u_2 & u_3 & u_4 & u_5 \\ u_3 & u_4 & u_5 & u_6 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

vanishes.

In view of the simplicity of the zeros, $f(z)+15\varepsilon z^2$ will have also real zeros, ε real and sufficiently small. The corresponding determinant will be

$$\Delta_\varepsilon = \begin{vmatrix} -1 & 0 & 1+\varepsilon & 0 \\ 0 & 1+\varepsilon & 0 & -1 \\ 1+\varepsilon & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}.$$

Adding the third row to the first row, and then the fourth column to the second column, we obtain

$$\Delta_\varepsilon = \begin{vmatrix} \varepsilon & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & -1 \\ 1+\varepsilon & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \varepsilon & 0 & \varepsilon \\ 0 & \varepsilon & 0 \\ 1+\varepsilon & 0 & -1 \end{vmatrix} = \varepsilon^2(-2-\varepsilon).$$

Hence Δ_ε is negative if ε is small enough, $\varepsilon \neq 0$.

The determinant associated with $f(z)+6\varepsilon z$ will be positive if $\varepsilon \neq 0$.

2. Turâns inequality ($l=2$) is not valid for general systems of orthogonal polynomials. We may expect a generalization of Turán's inequality in the following direction. Let $\alpha(x)$ be a distribution function with a spectrum on $x \geqq a$, and let $\{Q_n(x)\}$ be the associated system of

orthogonal polynomials normalized by the condition $Q_n(a) = 1$. Then the inequality $Q_{n-1}(x)\, Q_{n+1}(x) - [Q_n(x)]^2 < 0$ holds for all x on the spectrum.

We discuss this question by two different methods.

(a) The assertion is certainly not true in general, even if $\alpha(x)$ is absolutely continuous, $d\alpha(x) = w(x)\, dx$ and the spectrum is a finite interval. Indeed, we know that for the Legendre polynomials $P_n(x) = P_n$ the reverse inequality $P_{n-1} P_{n+1} - P_n^2 > 0$ holds for $x > 1$ [3]. Now let ε be an arbitrary positive number; we form the weight function

$$w(x) = w(\varepsilon\,;\, x) = \begin{cases} 1 & \text{for } \; -1 \leqq x \leqq 1\,, \\ \varepsilon & \text{for } \; 1+\varepsilon \leqq x \leqq 2\,, \end{cases}$$

and let $w(\varepsilon\,;\,x)$ be linear for $1 \leqq x \leqq 1 + \varepsilon$. The associated orthogonal polynomials, assuming the end value 1 at $x = -1$, approximate $(-1)^n P_n(x)$ uniformly in $-1 \leqq x \leqq 2$ as $\varepsilon \to 0$. This follows from the explicit formula for the orthogonal polynomials [13, (2.2.6)]. Thus for any $x > 1$, the opposite inequality holds provided ε is sufficiently small.

(b) Let us consider a system of polynomials defined recursively by the relations

$$Q_{n+1}(x) = \frac{\lambda_n + \mu_n - x}{\lambda_n}\, Q_n(x) - \frac{\mu_n}{\lambda_n}\, Q_{n-1}(x), \qquad n = 0, 1, 2, \ldots, \tag{3.1}$$

where $Q_{-1}(x) = 0$, $Q_0(x) = 1$, $\lambda_n > 0$, $\mu_n > 0$ for $n \geqq 0$ except $\mu_0 = 0$, It is known [Favard's theorem, cf. 7] that $Q_n(x)$ comprises a system of polynomials orthogonal with respect to a certain measure ψ whose spectrum is confined to the non-negative axis. In the form (3.1), the polynomials carry the natural normalization $Q_n(0) = 1$.

A direct calculation gives

$$U = Q_1^2 - Q_0 Q_2 = \frac{x^2}{\lambda_0}\left(\frac{1}{\lambda_0} - \frac{1}{\lambda_1}\right) + \frac{x}{\lambda_0 \lambda_1}(\mu_1 + \lambda_0 - \lambda_1). \tag{3.2}$$

We now set $\lambda_1 < \lambda_0$. Clearly for x sufficiently large U is negative while for x close to zero U is positive. This is true independent of the parameter values $\lambda_n, \mu_n, n \geqq 2$. We may select freely the parameter values $\lambda_n, \mu_n, n \geqq 2$, provided only they are positive. In particular, if λ_n and μ_n are chosen unbounded, then the spectrum of ψ is necessarily an unbounded set [see 7]. If also $\sum \frac{1}{\lambda_n \pi_n} = \infty$. and also $\sum \pi_n = \infty$ is satisfied where

$\pi_n = \dfrac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}$ (e.g., $\lambda_n = \mu_n = n$ for $n \geq 2$), then ψ has spectrum in every neighborhood of the origin while the origin is not a mass point [see 7].

Thus we see that the Turán inequality is not a general property of orthogonal polynomials, even restricting consideration only to the x values belonging to the spectrum of ψ. It appears that this inequality is intrinsically bound to the classical orthogonal polynomials in which category we include, in addition to the ultraspherical, Laguerre, and Hermite systems, also their discrete analogs.

3. Turán's inequality ($l = 2$) is not valid for general Jacobi polynomials. We follow the notation $P_n^{(\alpha,\beta)}(x)$ for the Jacobi polynomials used in [13, p. 58]. The inequality in question would be again

$$Q_{n-1}(x)\, Q_{n+1}(x) - [Q_n(x)]^2 < 0$$

where now $-1 < x < 1$, and

$$Q_n(x) = \frac{P_n^{(\alpha,\beta)}(x)}{\binom{n+\alpha}{n}}.$$

We set $x = -1$ so that the following inequality appears: $c_{n-1}\, c_{n+1} - c_n^2 < 0$ where

$$c_n = \frac{\binom{n+\beta}{n}}{\binom{n+\alpha}{n}}.$$

Let α be fixed and β be variable. The quantity $c_{n-1}\, c_{n+1} - c_n^2$ certainly vanishes for $\beta = \alpha$ and we shall prove that it changes sign for $\beta = \alpha$. Indeed,

$$\frac{c_n}{c_{n-1}} - 1 = \frac{n+\beta}{n+\alpha} - 1 = \frac{\beta-\alpha}{n+\alpha},$$

$$c_{n-1}\, c_n \left(\frac{c_{n+1}}{c_n} - \frac{c_n}{c_{n-1}}\right) = c_{n-1}\, c_n \left(\frac{\beta-\alpha}{n+\alpha+1} - \frac{\beta-\alpha}{n+\alpha}\right) = -\frac{(\beta-\alpha)\, c_{n-1}\, c_n}{(n+\alpha+1)(n+\alpha)}.$$

This makes the assertion obvious.

Of course, the opposite inequality $c_{n-1}\, c_{n+1} - c_n^2 > 0$ can not hold

unrestrictedly in $-1<x<1$ either, since the left hand side will be negative if $Q_{n-1}(x)=0$.

§ 4. Matrix multiplication, illustration. A theorem of Sylvester.

This section contains a few hints about matrices which will be useful in the later discussions.

1. How to prove the non-vanishing of a given determinant? Two principal methods will be used:

(a) We form the associated system of linear-homogeneous equations and show that it has only the trivial solution.

(b) We multiply the given determinant by certain non-vanishing determinants of known sign and discuss the resulting determinant. If the given determinant is real and symmetric, certain non-singular linear transformations of the associated quadratic form can be used.

Let us elaborate on (b). We consider the square (not necessary symmetric) matrices

$$A=(a_{pq}),\quad H=(h_{pq}),\qquad p,q=0,1,\ldots,l-1. \tag{4.1}$$

We have then the relations

$$B=H'AH=(b_{pq})\ ;\ b_{pq}=\Sigma h_{\mu p}\,a_{\mu\nu}\,h_{\nu q},\qquad \mu,\nu=0,1,\ldots,l-1. \tag{4.2}$$

We denote the corresponding determinants by $[A]$, $[B]$, $[H]$ so that $[B]=[A]\,[H]^2$ holds. In order to evaluate $[A]$ (or sgn $[A]$) we shall choose H in such a manner that $h_{pq}=0$ for $p>q$, moreover so that B should be a "simple" matrix (for instance a diagonal matrix or even the unit matrix). The relation permits then the evaluation of $[A]$. In particular, if all quantities involved are real and $h_{pp}\neq 0$ we shall have sgn $[A]=$ sgn $[B]$.

2. Illustration of (b). Let us evaluate the determinant

$$[A]=\begin{vmatrix} P_0 & P_1 & \ldots & P_{l-1} \\ P_1 & P_2 & \ldots & P_l \\ \cdot & \cdot & \ldots & \cdot \\ P_{l-1} & P_l & \ldots & P_{2l-2} \end{vmatrix} \tag{4.3}$$

where

$$P_n = P_n(x) = \frac{1}{\pi} \int_0^\pi (x + \sqrt{x^2-1}\cos\varphi)^n \, d\varphi \tag{4.4}$$

is Legendre's polynomial; (4.4) is the familiar Laplace's integral representation which will be a useful instrument [cf. (5.1)]. We assume $h_{\mu p} = 0$ for $\mu > p$ so that from (4.2)

$$\begin{aligned} b_{pq} = \frac{1}{\pi} \int_0^\pi & \sum_{\mu=0}^{p} h_{\mu p} (x + \sqrt{x^2-1}\cos\varphi)^\mu \\ & \cdot \sum_{\nu=0}^{q} h_{\nu q} (x + \sqrt{x^2-1}\cos\varphi)^\nu \, d\varphi . \end{aligned} \tag{4.5}$$

Here x is a parameter which we choose temporarily to be > 1; let $\sqrt{x^2-1}$ be positive. We determine H according to the condition

$$\sum_{\mu=0}^{p} h_{\mu p} (x + \sqrt{x^2-1}\cos\varphi)^\mu = \cos p\varphi , \quad p = 0, 1, \ldots, l-1 , \tag{4.6}$$

so that

$$h_{pp} (\sqrt{x^2-1})^p = 2^{p-1} , \qquad p = 1, 2, \ldots, l-1 . \tag{4.7}$$

For $p = 0$ the right-hand side must be 1 instead of $\frac{1}{2}$. Hence

$$b_{pq} = \frac{1}{\pi} \int_0^\pi \cos p\varphi \cos q\varphi \, d\varphi = \begin{cases} 0 \text{ for } p \neq q , \\ \frac{1}{2} \text{ for } p = q = 1, 2, \ldots, l-1 . \end{cases} \tag{4.8}$$

For $p = q = 0$ the number $\frac{1}{2}$ must be replaced by 1.

The result is the following identity:

$$\begin{aligned} [A] = [H]^{-2} [B] &= \prod_{p=0}^{l-1} (h_{pp}^{-2} b_{pp}) \\ &= \prod_{p=1}^{l-1} \left\{ (x^2-1)^p \, 2^{2-2p} \cdot \frac{1}{2} \right\} \\ &= 2^{-(l-1)^2} (x^2-1)^{l(l-1)/2} . \end{aligned} \tag{4.9}$$

3. Sylvester's theorem.

The following theorem of Sylvester [5, p. 31; cf. also 9] will be of great importance in the sequel.

Let A be a determinant of order l, and let

$$1 \leqq m_1 < m_2 \leqq l \quad , \quad 1 \leqq n_1 < n_2 \leqq l .$$

We denote by A_{mn} the determinant of order $l-1$ arising from A by striking out the row m and the column n. Similarly, we denote by $A\begin{Bmatrix} m_1 , m_2 \\ n_1 , n_2 \end{Bmatrix}$ the determinant arising from A by striking out the rows m_1 , m_2 and the columns n_1 , n_2. Then

$$(4.10) \qquad A \cdot A\begin{Bmatrix} m_1 , m_2 \\ n_1 , n_2 \end{Bmatrix} = \begin{vmatrix} A_{m_1 n_1} & A_{m_1 n_2} \\ A_{m_2 n_1} & A_{m_2 n_2} \end{vmatrix} .$$

Actually there exists a more general formulation of the identity of Sylvester but this is without consequence for our purposes.

§ 5. Orthogonal polynomials.

In this section, for further reference, we compile the definition and some basic properties of those orthogonal polynomials which will occur in the later text. Additional properties will be discussed if and when the necessity will arise.

1. Legendre and ultraspherical polynomials. Only two formulas should be mentioned involving Legendre polynomials:

$$(5.1) \quad P_n(x) = \frac{1}{\pi} \int_0^\pi (x + \sqrt{x^2-1} \cos \varphi)^n \, d\varphi , \quad \text{Laplace's integral, cf. (5.8)},$$

$$(5.2) \qquad (1-x^2)\, P'_n(x) = n\,(P_{n-1}(x) - xP_n(x)) , \qquad \text{cf. (5.6)}.$$

The following formulas on ultraspherical polynomials will be used:

$$(5.3) \quad P_n^{(\lambda)}(x) = \sum_{\nu=0}^{[\frac{n}{2}]} (-1)^\nu \frac{\Gamma(n-\nu+\lambda)}{\Gamma(\lambda)\,\Gamma(\nu+1)\,\Gamma(n-2\nu+1)} (2x)^{n-2\nu} = k_n^{(\lambda)} x^n + \dots ,$$

[13, (4.7.31)],

$$(5.4) \quad P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}, \quad k_n^{(\lambda)} = 2^n \binom{n+\lambda-1}{n}, \quad [13, (4.7.3), (4.7.9)],$$

$$(5.5) \qquad (1-x^2)\,y'' - (2\lambda+1)\,xy' + n\,(n+2\lambda)\,y = 0\,, \qquad y = P_n^{(\lambda)}(x)\,,$$

[13, (4.7.5)],

$$(5.6) \quad (1-x^2)\left(\frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)}\right)' = n\left(\frac{P_{n-1}^{(\lambda)}(x)}{P_{n-1}^{(\lambda)}(1)} - x\,\frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)}\right), \quad [13,\ (4.7.27)]\,,$$

$$(5.7) \quad \int_{-1}^{+1} [P_n^{(\lambda)}(x)]^2\,(1-x^2)^{\lambda-1/2}\,dx = h_n^{(\lambda)} = \pi^{1/2}\,\frac{\Gamma(\lambda+{}^1/_2)}{\Gamma(\lambda)}\,\frac{P_n^{(\lambda)}(1)}{n+\lambda}\,,$$

[13, (4.7.15)],

$$(5.8) \quad \frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)} = \pi^{-1/2}\,\frac{\Gamma(\lambda+{}^1/_2)}{\Gamma(\lambda)}\int_0^{\pi} (x+\sqrt{x^2-1}\cos\varphi)^n \sin^{2\lambda-1}\varphi\,d\varphi\,,$$ [4]

[13, (4.10.3)].

The special case $\lambda = \frac{1}{2}$ of (5.8) has been used in § 4.2. We mention also the formula for the "associated functions":

$$(5.9) \quad \frac{n!}{(n+\nu)!}\cdot(x^2-1)^{\nu/2}\left(\frac{d}{dx}\right)^{\nu} P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x+\sqrt{x^2-1}\cos\varphi)^n \cos\nu\varphi\,d\varphi.$$

$$\nu = 0, 1, \ldots, n.$$

This is a special (limiting) case of the following formula:

$$(5.10) \quad \Gamma(2\lambda)\cdot\frac{n!}{\Gamma(n+\nu+2\lambda)}\binom{\nu+2\lambda-2}{\nu}\cdot(x^2-1)^{\nu/2}\left(\frac{d}{dx}\right)^{\nu} P_n^{(\lambda)}(x)$$

$$= \pi^{-1/2}\,\frac{\Gamma(\lambda+{}^1/_2)}{\Gamma(\lambda)}\int_0^{\pi} (x+\sqrt{x^2-1}\cos\varphi)^n\,P_\nu^{(\lambda-1/2)}(\cos\varphi)\sin^{2\lambda-1}\varphi\,d\varphi$$

which is due to L. Gegenbauer. It follows from the addition theorem of the ultraspherical polynomials [cf. 2, Vol. 1, p. 177, (19)]. For $\nu = 0$, (5.10) yields (5.8).

In (5.7) we have $\lambda > -\frac{1}{2}$, in (5.8) and in (5.10) we have $\lambda > 0$.

---

4. We have

$$\lim_{\lambda\to 0} \lambda^{-1}\,P_n^{(\lambda)}(x) = \frac{2}{n}\,T_n(x)$$

(Tchebychev's polynomial, $n \geqq 1$).

2. Laguerre and Hermite polynomials.

(5.11) $$L_n^{(\alpha)}(x) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \frac{(-x)^\nu}{\nu!} = k_n^{(\alpha)}(-x)^n + \dots, \quad [13, (5.1.6)],$$

(5.12) $$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}, \quad k_n^{(\alpha)} = \frac{1}{n!}, \qquad [13, (5.1.7), (5.1.8)],$$

(5.13) $$xy'' + (\alpha+1-x)y' + ny = 0, \quad y = L_n^{(\alpha)}(x), \qquad [13, (5.1.2)],$$

(5.14) $$x\left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}\right)' = -x\frac{L_{n-1}^{(\alpha+1)}(x)}{L_n^{(\alpha)}(0)} = n\left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} - \frac{L_{n-1}^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(0)}\right),$$
[13, (5.1.14)],

(5.15) $$H'_n(x) = 2nH_{n-1}(x) = 2xH_n(x) - H_{n+1}(x), \qquad [13, (5.5.10)].$$

For the Laguerre polynomials we use sometime the alternative notation

(5.16) $$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} = Q_n^{(\alpha)}(x)$$

so that $Q_n^{(\alpha)}(0) = 1$; from the second part of (5.14):

(5.17) $$\frac{x}{\alpha+1} Q_n^{(\alpha+1)}(x) = Q_n^{(\alpha)}(x) - Q_{n+1}^{(\alpha)}(x).$$

3. Special polynomials of discrete measures.

(a) The Poisson-Charlier polynomials are defined by

(5.18) $$c_n(a\,;x) = c_n(x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu}\binom{x}{\nu}\frac{\nu!}{a^\nu}, \quad [13, (2.81.2)],$$

where $a>0$; $c_n(0) = 1$. They satisfy the symmetry relation

(5.19) $$c_n(x) = c_x(n), \quad n \text{ and } x = 0, 1, 2, \dots, \quad [2, \text{Vol. 2, p. 227, (7)}].$$

They are orthogonal with respect to the measure defined by the jump at x:

(5.20) $$j(x) = e^{-a}\frac{a^x}{x!}, \qquad x = 0, 1, 2, \dots$$

[Poisson's distribution; 13, (2.81.5)]. Finally

(5.21) $$c_n(a\,;x) = (-1)^n \frac{n!}{a^n} L_n^{(x-n)}(a), \qquad [13, (2.81.6)].$$

(b) Meixner's polynomials are defined in terms of the standard

hypergeometric functions as

$$(5.22) \qquad M_n(\beta, \gamma; x) = M_n(x) = F\left(-n, -x; \beta; 1-\frac{1}{\gamma}\right)$$

[2, Vol. 2, p. 225, (9)]

where $\beta > 0$, $0 < \gamma < 1$; $M_n(0) = 1$. They satisfy the symmetry relation

$$(5.23) \qquad M_n(x) = M_x(n), \qquad n \text{ and } x = 0, 1, 2, \ldots .$$

They are orthogonal with respect to the measure defined by the jump at x:

$$(5.24) \quad j(x) = (1-\gamma)^\beta \frac{(\beta)_x \gamma^x}{x!}, \quad (\beta)_0 = 1; \quad (\beta)_x = \beta(\beta+1)\ldots(\beta+x-1),$$

$$x = 0, 1, 2, 3, \ldots,$$

cf. loc. cit. p. 225, (12), (10), (11).

(c) Krawtchouk's polynomials are defined by

$$(5.25) \qquad k_n(N, p; x) = k_n(x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{N-x}{n-\nu} \binom{x}{\nu} p^{n-\nu} q^\nu,$$

[13, (2.82.2)],

where N is a positive integer, $p > 0$, $q > 0$, $p + q = 1$; $0 \leqq n \leqq N$. Writing $Q_n(x) = \dfrac{k_n(x)}{k_n(0)}$ we have the symmetry relation

$$(5.26) \qquad Q_n(x) = Q_x(n), \qquad n \text{ and } x = 0, 1, 2, \ldots, N.$$

A further symmetry relation is the following:

$$(5.27) \qquad k_n(N, p; x) = (-1)^n k_n(N, q; N-x), \qquad x = 0, 1, \ldots, N.$$

The polynomials $k_n(x)$ are orthogonal with respect to the measure defined by the jump at x:

$$(5.28) \qquad j(x) = \binom{N}{x} p^x q^{N-x}, \quad x = 0, 1, \ldots, N, \ [13, (2.82.1)]$$

(binomial distribution).

The definition (5.25) is meaningful for arbitrary integer n and for arbitrary x. If x is an integer, $0 \leqq x \leqq N$, all polynomials $k_n(x)$, $n \geqq N+1$, vanish.

(b) Tchebychev's polynomials of a discrete measure are defined by

(5.29) $$t_n(x) = n\,!\ \Delta^n \binom{x}{n}\binom{x-N}{n}, \qquad [13, (2.8.1)],$$

where N is a positive integer; $\Delta f(x) = f(x+1) - f(x)$. They are orthogonal with respect to a measure defined by the jump $j(x) = 1/N$; n and $x = 0, 1, \ldots, N-1$. We have the symmetry relation

(5.30) $$t_n(N-1-x) = (-1)^n t_n(x)\,,$$

and the "end values"

(5.31) $$(-1)^n t_n(0) = t_n(N-1) = n\,!\binom{N-1}{n}.$$

We are interested mainly in the case when n and x are integers, x varying from 0 to $N-1$. The definition (5.29) is meaningful generally. If x is an integer, $0 \leq x \leq N-1$, all polynomials $t_n(x)$, $n \geq N$, vanish.

In § 21 we shall consider briefly also a generalization $t_n(a, b; x)$ of Tchebychev's $t_n(x)$ where $a > -1$, $b > -1$. They are orthogonal with the jump

(5.32) $$j(x) = \frac{(\beta)_x(\gamma)_x}{x\,!\,(\delta)_x}, \qquad x = 0, 1, \ldots, N-1\,,$$

where $(\beta)_x$ etc. have the same meaning as in (5.24). Here

(5.33) $$\beta = 1 + a, \quad \gamma = 1 - N, \quad \delta = 1 - N - b\,.$$

There are interesting limit relations connecting $k_n(N, p; x)$ with the Hermite polynomials [13, (2.82.7)]., and $t_n(x)$ with the Legendre polynomials [13, (2.8.6)]. Also $t_n(a, b; x)$ is similarly related to the Jacobi polynomials. In addition we mention:

(5.34) $$H_n(x) = \lim_{\lambda\to\infty} (2\lambda^{1/2})^n \frac{P_n^{(\lambda)}(\lambda^{-1/2}x)}{P_n^{(\lambda)}(1)}, \qquad [13, (5.6.3)],$$

(5.35) $$Q_n^{(\beta-1)}(x) = \lim_{\gamma\to1} M_n\left(\beta, \gamma; \frac{\gamma x}{1-\gamma}\right), \quad [2, \text{Vol. 2, p. 226, (15)}];$$

hence the Hermite polynomials are limiting cases of the ultraspherical polynomials and the Laguerre polynomials are limiting cases of the Meixner polynomials. Finally we refer to the formula

(5.36) $$L_n^{(\alpha)}(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right), \qquad [13, (5.3.4)],$$

where on the right hand side Jacobi polynomials appear.

Chapter 2. DETERMINANTS OF THE WRONSKI TYPE

§ 6. Generalities on Wronskians.

All orthogonal polynomials $\{Q_n(x)\}$ occuring in this section will be normalized as in Theorem 1, i.e. $Q_n(x) = k_n(-x)^n + \ldots$, $k_n > 0$, or $Q_n(-\infty) = +\infty$. The order l of the determinants considered may be even or odd. We discuss here a few elementary properties of the Wronskians (1.2) and of the discrete Wronskians (1.3) defined in the Introduction. We use the symbol $W(n, l; x)$ as defined by (1.6).

1. Leading term of the Wronskian $W(n, l; x)$. This Wronskian is a polynomial in x; its highest term will be, apart from trivial positive constant factors, the following:

$$(-1)^{ln+\frac{l(l-1)}{2}} \cdot x^{ln}. \tag{6.1}$$

If the normalization is such that the highest term is $k_n x^n$, $k_n > 0$, the leading term will be x^{ln}.

For the proof of (6.1), we multiply the rows of (1.2) by $1, x, x^2, \ldots, x^{l-1}$ and the columns by $x^{-n}, x^{-n-1}, \ldots, x^{-n-l+1}$, respectively. For $x \to \infty$ we obtain, apart from positive constant factors, the limit

$$(-1)^{n+n+1+\ldots+n+l-1} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ n & n+1 & \ldots & n+l-1 \\ n(n-1) & (n+1)n & \ldots & (n+l-1)(n+l-2) \\ . & . & \ldots & . \\ n(n-1)\ldots(n-l+2) & . & \ldots & (n+l-1)(n+l-2)\ldots(n+1) \end{vmatrix}$$

$$= (-1)^{ln+\frac{l(l-1)}{2}} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ n & n+1 & \ldots & n+l-1 \\ n^2 & (n+1)^2 & \ldots & (n+l-1)^2 \\ . & . & \ldots & . \\ n^{l-1} & (n+1)^{l-1} & \ldots & (n+l-1)^{l-1} \end{vmatrix};$$

the second determinant arises from the first one by appropriate combination of the rows; it is a Vandermondian and its value is clearly positive. Hence the assertion follows.

We point out the following important consequences of (6.1). If x is large positive we have

$$(6.2)\qquad \operatorname{sgn} W(n, l; x) = \begin{cases} (-1)^{l/2} & \text{if } l \text{ even,} \\ (-1)^{n+\frac{l-1}{2}} & \text{if } l \text{ odd.} \end{cases}$$

If x is large negative we have

$$(6.3)\qquad \operatorname{sgn} W(n, l; x) = \begin{cases} (-1)^{l/2} & \text{if } l \text{ even,} \\ (-1)^{\frac{l-1}{2}} & \text{if } l \text{ odd.} \end{cases}$$

2. Discrete Wronskians. Let $\{Q_n(x)\}$ be associated with a discrete measure and normalized as in Theorem 3. In the sequel we shall use the notation (1.3) and in particular the symbols $u(n, l; r) = u_n(r)$; $r = 0, 1, 2, \ldots,$ defined by (1.11).

We have the following "mean value theorem":

$$(6.4)\quad Q\begin{pmatrix} n, n+1, \ldots, n+l-1 \\ x_0, x_1, \ldots, x_{l-1} \end{pmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \binom{x_0}{1} & \binom{x_1}{1} & \cdots & \binom{x_{l-1}}{1} \\ \cdot & \cdot & \cdots & \cdot \\ \binom{x_0}{l-1} & \binom{x_1}{l-1} & \cdots & \binom{x_{l-1}}{l-1} \end{vmatrix} \cdot$$

$$\cdot \begin{vmatrix} Q_n(\xi_0) & Q_{n+1}(\xi_1) & \cdots & Q_{n+l-1}(\xi_{l-1}) \\ Q'_n(\xi_0) & Q'_{n+1}(\xi_1) & \cdots & Q'_{n+l-1}(\xi_{l-1}) \\ \cdot & \cdot & \cdots & \cdot \\ Q_n^{(l-1)}(\xi_0) & Q_{n+1}^{(l-1)}(\xi_1) & \cdots & Q_{n+l-1}^{(l-1)}(\xi_{l-1}) \end{vmatrix}$$

where $x_0 < x_1 < \ldots < x_{l-1}$ and $\xi_1, \ldots, \xi_{l-1}$ denote certain real numbers satisfying the inequalities

$$(6.5)\qquad \xi_0 = x_0\,,\ \xi_0 < \xi_1 < x_1\,,\ \xi_1 < \xi_2 < x_2\,,\ \ldots,\ \xi_{l-2} < \xi_{l-1} < x_{l-1}\,.$$

We refer to [11, Vol. 2, Chapter 5, Problem 96, p. 54]. The first determinant is, apart from trivial positive factors, a Vandermondian, cf. **1**. Hence the sign of the discrete Wronskian (6.4) is the same as that of the second determinant occuring on the right. We note the following consequences.

$$(6.6)\qquad \operatorname{sgn} Q\begin{pmatrix} 0, 1, \ldots, l-1 \\ x_0, x_1, \ldots, x_{l-1} \end{pmatrix} = (-1)^{\frac{l(l-1)}{2}}\,;\quad x_0 < x_1 < \ldots < x_{l-1}\,.$$

(Of course this can be proved also directly by combining the columns).

Moreover let $x_0 \to +\infty$ so that all variables $x_k \to +\infty$; we have then:

(6.7) $$\lim_{x_0 \to +\infty} \operatorname{sgn} Q\begin{pmatrix} n, n+1, \ldots, n+l-1 \\ x_0, x_1, \ldots, x_{l-1} \end{pmatrix} = (-1)^{ln+\frac{l(l-1)}{2}};$$

$$x_0 < x_1 < \ldots, < x_{l-1}.$$

This follows from (6.4) proceeding similarly as in **1**.

3. Finally we mention the following important information:

(6.8) $$\operatorname{sgn} Q\begin{pmatrix} n, n+1, \ldots, n+l-1 \\ a_0, a_1, \ldots, a_{l-1} \end{pmatrix} = \begin{cases} (-1)^{l/2} & \text{if } l \text{ even,} \\ (-1)^{\frac{l-1}{2}} & \text{if } l \text{ odd.} \end{cases}$$

This Wronskian is actually the first term $u(n, l; 0) = u_n(0)$ of the sequence $\{u_n(r); r = 0, 1, 2, \ldots\}$ occuring in Theorem 4.

The first part of (6.8) is in fact a special case of Theorem 3 to be proved in § 8, cf. (1.12). Let us consider the second case, l odd. We shall prove the sharper fact:

(6.9) $$\operatorname{sgn} Q\begin{pmatrix} n, n+1, \ldots, n+l-1 \\ x, a_1, \ldots, a_{l-1} \end{pmatrix} = (-1)^{\frac{l-1}{2}}, \quad l \text{ odd}, \ x \leqq a_0.$$

Here we shall make use of Theorem 3. This sharper inequality is definitely not true for even l as can be seen already for $l = 2$.

For the proof of (6.9) we note that the expression on the left of (6.9) is a polynomial $f(x)$ of degree $u+l-1$ which is a linear combination of the polynomials $Q_n(x), Q_{n+1}(x), \ldots, Q_{n+l-1}(x)$. The coefficients of $Q_n(x)$ and $Q_{n+l-1}(x)$ are not zero [cf. (1.12), l even]. Suppose that $f(x_0) = 0$ for some x_0, $x_0 \leqq a_0$; we conclude that

$$f(x_0) = 0, f(a_1) = 0, \ldots, f(a_{l-1}) = 0,$$

so that

$$f(x) = (x-x_0)(x-a_1) \ldots (x-a_{l-1})\varphi(x)$$

where $\varphi(x)$ is a polynomial of degree $n-1$ not identically zero. Now, in view of the orthogonality of the $Q_n(x)$,

$$0 = \int_{a_0}^{\infty} f(x)\varphi(x)\,d\alpha(x) = \int_{a_0}^{\infty} (x-x_0)(x-a_1)\ldots(x-a_{l-1})[\varphi(x)]^2\,d\alpha(x)$$

$$= \sum_{r=0}^{\infty} (a_r-x_0)(a_r-a_1)\ldots(a_r-a_{l-1})[\varphi(a_r)]^2 \cdot j_r$$

where $j_r > 0$ is the jump of the distribution function $\alpha(x)$ at $x = a_r$. In the right hand sum we may suppress the terms $r = 1, 2, \ldots, l-1$. The term $r = 0$ is of the sign $(-1)^{l-1} = 1$ and the terms $r \geqq l$ are also non-negative and not all zero which is a contradiction. Hence (6.9) has a constant sign for $x \leqq a_0$. Letting $x \to -\infty$ we obtain the term with $Q_{n+l-1}(x)$ as the leading one; its algebraic complement is, according to Theorem 3, of the sign $(-1)^{(l-1)/2}$.

§ 7. Theorems 1 and 2, general measure.

The purpose of this section is to prove Theorems 1 and 2 formulated in the Introduction. In these Theorems l is even and odd, respectively.

1. In order to prove Theorem 1, l even, we assume that $W(n, l; x) = 0$ holds for some real $x = x_0$; we conclude the existence of certain real constants $\lambda_0, \lambda_1, \ldots, \lambda_{l-1}$ not all zero such that the polynomial

$$f(x) = \lambda_0 Q_n(x) + \lambda_1 Q_{n+1}(x) + \ldots + \lambda_{l-1} Q_{n+l-1}(x) \tag{7.1}$$

satisfies the following equations:

$$f(x_0) = 0, \, f'(x_0) = 0, \ldots, f^{(l-1)}(x_0) = 0.$$

Hence $f(x)$ would have a zero of multiplicity at least l at x_0. On the other hand, in view of the orthogonality property we have

$$\int_{-\infty}^{\infty} f(x)\, q(x)\, d\alpha(x) = 0$$

where $q(x)$ is any polynomial of degree $n-1$. In the usual way [cf, 13, p. 44] we conclude that $f(x)$ must have at least n nodal zeros.

If x_0 is different from all these nodal zeros, we have found in the whole $n + l$ zeros which is a contradiction. If x_0 coincides with one of these nodal zeros, the multiplicity of x_0 must be odd, i.e. at least $l + 1$. Thus we have located at least $n - 1 + l + 1$ zeros which is again a contradiction.

We proved that $W(n, l; x)$ keeps a constant sign for all real x; now we use (6.2) or (6.3), l even. This establishes the proof of Theorem 1.

2. The simple argument used above permits also to obtain information about the sign of the Wronskian $W(n, l; x)$ for odd l provided the spectrum is in a finite or half-infinite interval and x is situated outside of the spectrum.

Let l be odd. We assume that $\alpha(x)$ is constant for $x \leq a$ so that the spectrum will lie in $[a, +\infty)$. The previous argument yields that $W(n, l; x)$ can not vanish for any $x = x_0 < a$; assuming the contrary, the function $f(x)$ formed as in (7.1) would have again n nodal zeros which are in this case all to the right of a hence different from x_0. This yields $n+l$ zeros, i.e. a contradiction. Now for $x \to -\infty$ we have (6.3) so that

$$\operatorname{sgn} W(n, l; x) = (-1)^{\frac{l-1}{2}}, \qquad l \text{ odd, } x \leq a. \tag{7.2}$$

In a similar fashion we can show that if $\alpha(x)$ is constant for $x \geq b$, we have

$$\operatorname{sgn} W(n, l; x) = (-1)^{n+\frac{l-1}{2}}, \qquad l \text{ odd, } x \geq b. \tag{7.3}$$

3. Now we proceed to the proof of Theorem 2, l odd. In order to show the Sturm character of the set $W(n, l; x)$, $n = 0, 1, 2, \ldots, x$ real, we follow the instructions of § 2.1. For $n = 0$ we have

$$W(0, l; x) = Q_0(x) Q'_1(x) \ldots Q^{(l-1)}_{l-1}(x), \quad \operatorname{sgn} W(0, l; x) = (-1)^{\frac{l-1}{2}}. \tag{7.4}$$

The key formula from which our deductions flow, is obtained by invoking the Sylvester identity (4.10). We consider $W(n-1, l+1; x)$ and strike out one or both of the last two rows as well as of the first and last columns. In consequence,

$$\begin{aligned} &W(n-1, l+1; x) \cdot W(n, l-1; x) \\ &= \begin{vmatrix} W(n-1, l; x) & W(n, l; x) \\ DW(n-1, l; x) & DW(n, l; x) \end{vmatrix}, \quad D = \frac{d}{dx}. \end{aligned} \tag{7.5}$$

By Theorem 1 the left hand side is strictly negative for all real x. It follows that

$$W(n-1, l; x) \cdot DW(n, l; x) < 0 \quad \text{when} \quad W(n, l; x) = 0 \tag{7.6}$$

and

$$W(n, l; y) \cdot DW(n-1, l; y) > 0 \quad \text{when} \quad W(n-1, l; y) = 0. \tag{7.7}$$

These relations yield the properties asserted for the zeros of $W(n, l; x)$;

cf. § 2.1. We have to take also the second part of (6.2) and the second part of (6.3) into account.

If the spectrum is restricted to a finite or half infinite interval we infer the same Sturm character in the respective interval. We have to take the remarks of 2 (see above) into account.

§ 8. Theorem 3, discrete measure, l even.

1. The proof of Theorem 3 is based on induction in a definite sense to be explained below. We denote generally by (a, k) a sequence of the element a of the spectrum repeated k times. Let $(x) = (x_0, x_1, \ldots, x_{l-1})$ be defined by the following sequence in the order indicated:

$$(8.1) \qquad (x) : (a_r, \nu_r)(a_{r+1}, \nu_{r+1}) \ldots (a_s, \nu_s), \; \nu_r + \nu_{r+1} + \ldots + \nu_s = l,$$

where $\nu_r, \nu_{r+1}, \ldots, \nu_s$ are all positive except possibly $\nu_s = 0$. (The condition $\nu_s = 0$ means that a_s actually does not occur. Such a sequence can always be "glued on" if we care provided that $\nu_{s-1} > 0$). The induction is based on the elementary step replacing (x) by (x') where

$$(8.2) \quad (x') : (a_r, \nu_r)(a_{r+1}, \nu_{r+1}) \ldots (a_{s-2}, \nu_{s-2})(a_{s-1}, \nu_{s-1} - 1)(a_s, \nu_s + 1).$$

Here $\nu_{s-1} \geqq 2$, $\nu_s \geqq 0$. With the aid of such steps the transition from (a_r, l) to any sequence (x) can be performed as follows:

(a_r, l);

$(a_r, l-1)(a_{r+1}, 1)$; $(a_r, l-2)(a_{r+1}, 2)$; ...; $(a_r, \nu_r)(a_{r+1}, l-\nu_r)$;

$(a_r, \nu_r)(a_{r+1}, l-\nu_r-1)(a_{r+2}, 1)$; ...; $(a_r, \nu_r)(a_{r+1}, \nu_{r+1})(a_{r+2}, l-\nu_r-\nu_{r+1})$;

. . . . . .

The induction consists of assuming that (1.9) holds for the sequence (x) and proving the same for the sequence (x'). We point out (1.10) which corresponds to $(x) = (a_r, l)$.

2. We consider the expression on the left of (1.9) where (x) has been replaced by (x'), that is

$$(-1)^{\frac{l}{2}} Q\begin{pmatrix} n, n+1, \ldots, n+l-1 \\ x'_0, x'_1, \ldots, x'_{l-1} \end{pmatrix}.$$

We form a polynomial $f(x)$ of degree $n + l - 1$ by replacing the last element x'_{l-1} of (x') by the continuous variable x; this polynomial is a

linear combination of $Q_n(x), Q_{n+1}(x), \dots, Q_{n+l-1}(x)$, and we have in view of (8.2)

$$(8.3)\quad \begin{aligned} &f(a_h) = f'(a_h) = \dots = f^{(\nu_h-1)}(a_h) = 0, \quad h = r, r+1, \dots, s-2, s, \\ &f(a_{s-1}) = f'(a_{s-1}) = \dots = f^{(\nu_{s-1}-2)}(a_{s-1}) = 0 . \end{aligned}$$

Hence

$$(8.4)\quad f(x) = (x-a_r)^{\nu_r} \dots (x-a_{s-2})^{\nu_{s-2}} (x-a_{s-1})^{\nu_{s-1}-1} (x-a_s)^{\nu_s} \varphi(x) = f_1(x)\varphi(x)$$

where $\varphi(x)$ is of degree n. The induction hypothesis implies that

$$(8.5)\quad (-1)^{\nu_s} f^{(\nu_{s-1}-1)}(a_{s-1}) > 0 .$$

To bring the induction to completion we need to prove that

$$(8.6)\quad f^{(\nu_s)}(a_s) > 0 .$$

In view of (8.4) the inequality (8.5) means that $\varphi(a_{s-1}) > 0$; similarly (8.6) means that $\varphi(a_s) > 0$. Now suppose to the contrary that $\varphi(a_s) \leqq 0$. Then $\varphi(x)$ must have at least one zero β in

$$a_{s-1} < x \leqq a_s , \quad \varphi(x) = (x-\beta)\,\varphi_1(x) .$$

By using the orthogonality:

$$\int_{-\infty}^{\infty} f_1(x)\,(x-\beta)\,[\varphi_1(x)]^2\, d\alpha(x) = 0 .$$

The polynomial $f_1\ x)(x-\beta)$ is of degree l; it is zero for $x = a_r, \dots, a_{s-1}$, it is non-negative for $x = a_s$, and positive for $x = a_i$ where $i > s$ or $i < r$ (l even). Evaluating the integral we observe the contradiction.

This establishes the assertion of Theorem 3.

3. Remark. In this induction procedure, we used only the following two properties of the polynomials $Q_m(x)$:

(a) $Q_m(x) = k_m(-x)^m + \dots, k_m > 0$;

(b) $\displaystyle\int_{-\infty}^{\infty} Q_m(x)\, x^\rho\, d\alpha(x) = 0, \ \rho < n; \qquad m = n, n+1, \dots, n+l-1 .$

With other words we did not use the orthogonality of the system $\{Q_m(x)\}$ fully, only the orthogonality of the polynomials $Q_n(x), \dots, Q_{n+l-1}(x)$ to every x^ρ.

Hence we proved in fact the following generalization of Theorem 3: Let $\{q_m(x)\}$ be any system of polynomials satisfying the two conditions which arise from (a), (b) by replacing $Q_m(x)$ by $q_m(x)$; let l be even. Then on the same spectrum as before (i.e. x_ν have the same meaning as before)

$$(-1)^{l/2}\,[q_{n+\mu}(x_\nu)]_0^{l-1} > 0\,. \tag{8.7}$$

In this determinant μ and ν run from 0 to $l-1$.

However, this is a trivial consequence of Theorem 3 since under the conditions mentioned:

$$q_{n+\mu}(x) = \sum_{k=0}^{\mu} \alpha_{\mu k}\, Q_{n+k}(x)\,, \qquad \mu = 0, 1, \ldots, l-1$$

with certain real constants $\alpha_{\mu k}$, $\alpha_{\mu\mu} > 0$, so that the following matrix equation holds:

$$(q_{n+\mu}(x_\nu)) = (\alpha_{\mu\nu})\,(Q_{n+\mu}(x_\nu))\,.$$

Passing to the determinants, (8.7) will be obvious.

This remark will be useful in the proof of Theorem 8, (b) and (c), cf. § 22.

§ 9. Theorem 4, discrete measure, l odd.

1. For the proof of Theorem 4 we note first that the sequence $\{u(0, l; r) = u_0(r);\ r = 0, 1, 2, \ldots\}$ has no sign variations,

$$\operatorname{sgn} u_0(r) = (-1)^{(l-1)/2}\,,$$

cf. (6.6). Also $\operatorname{sgn} u_n(0) = (-1)^{(l-1)/2}$ according to (6.8), and $u_n(r)$ has the sign $(-1)^{n+(l-1)/2}$ for large r, cf. (6.7).

The proof resembles that of Theorem 2. Its validity leans heavily on Theorem 3. A suitable application of Sylvesters identity (4.10) yields:

$$\begin{aligned} u(n, l-1; r)\, u(n-1, l+1; r-1) &= \begin{vmatrix} u(n-1, l; r-1) & u(n, l; r-1) \\ u(n-1, l; r) & u(n, l; r) \end{vmatrix} \\ &= \begin{vmatrix} u_{n-1}(r-1) & u_n(r-1) \\ u_{n-1}(r) & u_n(r) \end{vmatrix} \end{aligned} \tag{9.1}$$

where the sequences $\{u(n, l; r) = u_n(r)\}$ on the right are defined by the

determinant (1.2) of order l whereas the sequences on the left have the orders $l-1$ and $l+1$, respectively. By Theorem 3 the left hand is n e g a t i v e so that we have the fundamental inequality

$$(9.2) \qquad \begin{vmatrix} u_{n-1}(r-1) & u_n(r-1) \\ u_{n-1}(r) & u_n(r) \end{vmatrix} < 0 .$$

This shows that no two consecutive members of the sequences

$$\{u_n(r)\,;\, r = 0, 1, 2, \ldots\}$$

can vanish, i.e. the associated function $u_n(x)$, arising by linear interpolation, has only nodal zeros and no nodal intervals.

2. Let α and β be two successive sign variations in the sequence $\{u_{n-1}(r)\,;\, r = 0, 1, 2, \ldots\}$, $\alpha < \beta$. Let us assume that

$$(9.3) \qquad \begin{cases} u_{n-1}(\alpha) = A > 0\,, & u_{n-1}(\alpha-1) = -B \leqq 0\,; \\ u_{n-1}(\beta) = -D < 0\,, & u_{n-1}(\beta-1) = C \geqq 0\,. \end{cases}$$

Hence there is a nodal zero of $u_{n-1}(x)$ at $\alpha-1$ if $B = 0$ and one in the open interval $(\alpha-1, \alpha)$ if $B > 0$; similarly for $C = 0$ and $C > 0$. Writing $r = \alpha$ and $r = \beta$ in (9.2) we obtain

$$(9.4) \qquad A\,u_n(\alpha-1) + B\,u_n(\alpha) > 0\,, \quad C\,u_n(\beta) + D\,u_n(\beta-1) < 0\,.$$

We conclude that at least one of the numbers $u_n(\alpha-1)$, $u_n(\alpha)$ must be positive and at least one of the numbers $u_n(\beta-1)$, $u_n(\beta)$ must be negative. This yields a nodal zero of $u_n(x)$ between $\alpha-1$ and β. We must make this assertion more precise by showing that $u_n(x)$ has at least one zero between the zeros of $u_{n-1}(x)$ mentioned above.

Assuming first $B = 0$ [so that $u_{n-1}(x)$ has a zero at $\alpha-1$] we see from (9.4) that $A\,u_n(\alpha-1) > 0$, hence $u_n(x)$ has a zero $> \alpha-1$. Further let $B > 0$ [$u_{n-1}(x)$ has a zero in $(\alpha-1, \alpha)$]. Now if $u_n(\alpha) = 0$, $u_n(x)$ has a zero at α; and if $u_n(\alpha) > 0$, it has a zero $> \alpha$. Thus the only interesting case to assume is the following:

$$u_n(\alpha) = -A' < 0\,, \quad u_n(\alpha-1) = B' > 0\,, \quad AB' - A'B > 0\,, \quad A, B, A', B' > 0\,.$$

In this case both $u_{n-1}(x)$ and $u_n(x)$ have a zero in $(\alpha-1, \alpha)$, say $\alpha-1+p$ and $\alpha-1+p'$, respectively. Since

$$\frac{p}{B} = \frac{1-p}{A}\,, \quad \frac{p'}{B'} = \frac{1-p'}{A'}$$

we conclude that

$$p = \frac{B}{A+B} < p' = \frac{B'}{A'+B'}$$

so that the zero of $u_n(x)$ is to the right of that of $u_{n-1}(x)$ (in the strong sense). We show in a quite similar fashion the existence of a zero of $u_n(x)$ to the left of that of $u_{n-1}(x)$.

Between two zeros of $u_n(x)$ we can locate one zero of $u_{n-1}(x)$.

As to the existence of the proper number of zeros we take the instructions of § 2.1 into account; cf. the remark above concerning $\operatorname{sgn} u_n(x)$ for $x = 0$ and $x \to \infty$.

§ 10. An induction process. Applications.

The following counterpart of a theorem of Christoffel [13, pp. 29—31] is valid for distributions of the discrete type. Let

$$\{Q_n(x) = k_n(-x)^n + \dots, \; k_n > 0\}$$

be the system of orthogonal polynomials associated with a distribution the spectrum of which consists of the points $a_0 < a_1 < a_2 < \dots$. Let l be even. If $b_1, b_2, \dots, b_l$ denote a system of "successive" points of the spectrum, the polynomials $\{G_n(x)\,;\, n = 0, 1, 2, \dots\}$ defined by

$$Q\begin{pmatrix} n, n+1, \dots, n+l \\ x, b_1, \dots, b_l \end{pmatrix} = (-1)^{l/2}(x-b_1)\dots(x-b_l)\,G_n(x) \tag{10.1}$$

are orthogonal with respect to a distribution whose spectrum is the same as the original one except for the points $b_1, b_2, \dots, b_l$ which have to be discarded. Several of the b_i may coincide, cf. the explanation to Theorem 3; the masses of $Q_n(x)$ and $G_n(x)$ are, of course, generally different. According to Theorem 3

$$(-1)^{l/2}\,Q\begin{pmatrix} n, \dots, n+l-1 \\ b_1, \dots, b_l \end{pmatrix} > 0$$

so that the coefficient of x^n in $G_n(x)$ is different from 0; it is again of the sign $(-1)^n$.

The proof follows the same line as loc. cit. The left hand side of (10.1) is divisible by $(x-b_1)(x-b_2)\dots(x-b_l)$ and the latter polynomial

is positive on the spectrum except for the points b_i where it is zero. The orthogonality property is obvious.

Several applications of this simple remark will be presented.

1. Another arrangement for the proof of Theorem 3. The following argument is also of inductive character but slightly simpler than that of § 8.

We assume that l is even and the theorem is already established for determinants of the size $(l-2) \times (l-2)$. We aim at the proof of the inequality

$$(-1)^{l/2} Q\begin{pmatrix} n, n+1, \ldots, n+l-1 \\ x_0, x_1, \ldots, x_{l-1} \end{pmatrix} > 0 \tag{10.2}$$

where $x_0, x_1, \ldots, x_{l-1}$ are "successive" points of the spectrum. We define the polynomials $\{G_n(x)\}$ by

$$Q\begin{pmatrix} n, n+1, \ldots, n+l-2 \\ x, x_2, \ldots, x_{l-1} \end{pmatrix} = (-1)^{(l-2)/2}(x-x_2)\ldots(x-x_{l-1})\, G_n(x)\,. \tag{10.3}$$

The highest coefficient of $G_n(x)$ is different from zero, by the induction hypothesis. It has the sign $(-1)^n$. Now we apply Sylvester's theorem to the determinant on the left on (10.2) striking out the two first rows and the first and last columns; we obtain

$$\begin{aligned} &Q\begin{pmatrix} n, n+1, \ldots, n+l-1 \\ x_0, x_1, \ldots, x_{l-1} \end{pmatrix} \cdot Q\begin{pmatrix} n+1 \ldots, n+l-2 \\ x_2, \ldots, x_{l-1} \end{pmatrix} \\ &= \begin{vmatrix} Q\begin{pmatrix} n+1, \ldots, n+l-1 \\ x_1, x_2, \ldots, x_{l-1} \end{pmatrix} & Q\begin{pmatrix} n, \ldots, n+l-2 \\ x_1, x_2, \ldots, x_{l-1} \end{pmatrix} \\ Q\begin{pmatrix} n+1, \ldots, n+l-1 \\ x_0, x_2, \ldots, x_{l-1} \end{pmatrix} & Q\begin{pmatrix} n, \ldots, n+l-2 \\ x_0, x_2, \ldots, x_{l-1} \end{pmatrix} \end{vmatrix}. \end{aligned} \tag{10.4}$$

By the induction hypothesis the second factor on the left is of the sign $(-1)^{(l-2)/2}$; we have to show that the 2×2 determinant on the right is negative.

The factor of $G_n(x)$ in (10.3) is of the sign $(-1)^{(l-2)/2}$ for $x = x_0$ and $x = x_1$; Thus the determinant on the right of (10.4) is, apart from trivial positive factors, equal to

$$\begin{vmatrix} G_{n+1}(x_1) & G_n(x_1) \\ G_{n+1}(x_0) & G_n(x_0) \end{vmatrix} = G\begin{pmatrix} n, n+1 \\ x_0, x_1 \end{pmatrix} \tag{10.5}$$

which can be treated as the special case $l=2$ of the theorem. If $x_0 = x_1$ we obtain a Wronskian which is trivially negative, by Theorem 1. If $x_0 < x_1$, we follow essentially again the argument of § 8 but now in a much simpler setting. We consider the polynomial

$$G\begin{pmatrix} n, & n+1 \\ x, & x_1 \end{pmatrix} = f(x) = (x - x_1)\varphi(x).$$

We know that $\varphi(x_1) > 0$ [cf. 13, (3.2.4)], and we have to show that $\varphi(x_0) > 0$. Let $\varphi(x_0) \leqq 0$ so that $\varphi(x) = 0$ for some β,

$$x_0 \leqq \beta < x_1 \; , \; f(x) = (x - x_1)(x - \beta)\varphi_1(x).$$

In view of the orthogonality:

$$\int_{-\infty}^{\infty} (x - x_1)(x - \beta)[\varphi_1(x)]^2 \, d\alpha(x) = 0$$

and the contradiction follows in the same way as in § 8.

2. A generalization of Theorem 3. The previous argument requires only a slight modification in order to yield the following more general

Theorem 3′: Let l be even. Inequality (10.2) holds provided the set $x_0, x_1, \ldots, x_{l-1}$ consists of "blocks" of successive elements of the spectrum each block containing an even number of elements and distinct blocks do not contain common elements.

We may choose, for instance, the successive elements $a_1, \ldots, a_\alpha$ for the first block, α even, then the successive elements $b_1, \ldots, b_\beta$ for the second block, β even, etc. $a_\alpha < b_1, \ldots$. Again, the polynomial

$$(x - a_1) \ldots (x - a_\alpha) \; , \; (x - b_1) \ldots (x - b_\beta), \ldots$$

will be non negative on the spectrum so that this case can be reduced to determinant of size 2×2 as above.

3. A dual to Theorem 4.

Theorem 4′: With the same notation and the same assumptions as in Theorem 4, the sequences

(10.6) $\{u_n(r)_{\cdot}; n = 0, 1, 2, \ldots\}, \qquad r = 0, 1, 2, \ldots,$

form a Sturm set provided l is odd.

The dual character of this theorem to Theorem 4 is obvious. The sequence (10.6) of "number" r will have exactly r sign changes. The function $u_x(r)$ arising from $u_n(r)$ by the usual linear interpolation with respect to n, will have exactly r nodal points and no nodal intervals, i.e. $u_n(r)$ and $u_{n-1}(r)$ can not vanish simultaneously. The nodal points of the functions $u_x(r)$ and $u_x(r+1)$ strictly interlace.

We verify easily the usual conditions for Sturm sets. We have

$$(10.7) \qquad \begin{aligned} \operatorname{sgn} u_0(r) &= (-1)^{(l-1)/2}, \qquad &&\text{by (6.6)},\\ \operatorname{sgn} u_n(0) &= (-1)^{(l-1)/2}, \qquad &&\text{by (6.8)}. \end{aligned}$$

From (9.2) we conclude that $u_{n-1}(r)$ and $u_n(r)$ can not vanish at the same time. Also, in view of the complete symmetry of (9.2) with respect of n and r we can use the same argument as in § 9.2, interchanging n and r. We find that between two consecutive sign variations of

$$\{u_n(r-1)\ ;\ n = 0, 1, 2, \ldots\}$$

there is one sign variation of $\{u_n(r)\ ;\ n = 0, 1, 2, \ldots\}$ and conversely.

Finally we need information about the sign of $u_n(r)$ for large n. For this purpose we use the construction (10.1) forming

$$(10.8) \qquad Q\begin{pmatrix} n, n+1, \ldots, n+l-1 \\ x, a_{r+1}, \ldots, a_{r+l-1} \end{pmatrix} = (-1)^{\frac{l-1}{2}} (x - a_{r+1}) \ldots (x - a_{r+l-1})\, G_n(x)$$

where $G_n(x)$ is orthogonal with respect to a measure with the spectrum $a_0, a_1, \ldots, a_r, a_{r+l}, a_{r+l+1}, \ldots$. Moreover

$$u_n(r) = (-1)^{\frac{l-1}{2}} (a_r - a_{r+1}) \ldots (a_r - a_{r+l-1})\, G_n(a_r).$$

Now [cf. Appendix, § 31] we have if n is sufficiently large, $\operatorname{sgn} G_n(a_r) = (-1)^r$ so that $\operatorname{sgn} u_n(r) = (-1)^{r+\frac{l-1}{2}}$.

These facts are sufficient to establish the assertion.

The theorem just proved can also be generalized in a similar fashion as in **2**. We take, as before, blocks of even length each one arranged in the natural order and we denote by $\{a'_r\}$ the remaining elements of the spectrum. In addition to the blocks we consider the elements $a'_r, a'_{r+1}, \ldots, a'_{r+l-1}$ where l is a fixed odd number; we order the total set of elements in the

natural way. The resulting determinants form again a Sturm set if r ranges from 0 to ∞.

§ 11. Application to the Poisson-Charlier and Laguerre polynomials.

We apply the previously proved Theorem 3 to the special case of the Poisson-Charlier polynomials. An interesting inequality emerges which yields, after suitable transformations, Theorem 5 in one special case, namely for the Laguerre polynomials $L_n^{(\alpha)}(x)$ with integer α, $\alpha = 0, 1, 2, \ldots$.

1. We denote by $c_n(a\,;x) = c_n(x)$ the Poisson-Charlier polynomials normalized as in (5.18) so that $c_n(0) = 1$. The highest coefficient is of the sign $(-1)^n$. Further we denote by $Q_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) / L_n^{(\alpha)}(0)$ the Laguerre polynomials normalized by the condition $Q_n^{(\alpha)}(0) = 1$ [cf. Theorem 5 and (5.16)] where the upper index α indicates parameters and not derivatives. We shall make use of the important relation (5.21) which we write in the form

$$c_n(a\,;x) = (-1)^n a^{-n} n! \binom{x}{n} Q_n^{(x-n)}(a). \tag{11.1}$$

Assuming that l is even, we apply Theorem 3 to the functions $Q_n(x) = c_n(a\,;x)$ choosing $x_0 = \alpha$, $x_1 = \alpha + 1, \ldots, x_{l-1} = \alpha + l - 1$ where α is any integer, $\alpha \geqq n + l - 1$. Thus we have the determinantal inequality

$$(-1)^{l/2} \left[c_{n+\mu}(a\,;\alpha+\nu)\right]_0^{l-1} > 0. \tag{11.2}$$

In all determinants occuring in this section the indices μ, ν designating the rows and columns, run from 0 to $l-1$.

We transcribe inequality (11.2) into one involving Laguerre polynomials by making use of (11.1); we obtain

$$(-1)^{\frac{l}{2}} \left[(-1)^{n+\mu} a^{-n-\mu}(n+\mu)! \binom{\alpha+\nu}{n+\mu} Q_{n+\mu}^{(\alpha+\nu-n-\mu)}(a)\right]_0^{l-1} > 0,$$

or taking out trivial factors:

$$\left[\frac{Q_{n+\mu}^{(\alpha+\nu-n-\mu)}(a)}{(\alpha+\nu-n-\mu)!}\right]_0^{l-1} > 0. \tag{11.3}$$

We write here x instead of a and replace μ by $l-1-\mu$ which amounts to reversing the order of the rows. [This means multiplication by

$(-1)^{l/2}]$. At the same time we replace $\alpha-n-l+1$ by α so that the following inequality results:

$$(11.4)\qquad (-1)^{\frac{l}{2}}\left[\frac{Q^{(\alpha+\nu+\mu)}_{n+l-1-\mu}(x)}{(\alpha+\nu+\mu)!}\right]_0^{l-1} > 0\,.$$

It holds for $\alpha = 0, 1, 2, \ldots$.

2. Now we use the identity (5.17) which we generalize as follows:

$$(11,5)\qquad \frac{Q^{(\alpha)}_{n+r}(x)}{\alpha!} = \sum_{k=0}^{r}\binom{r}{k}(-x)^k\frac{Q^{(\alpha+k)}_n(x)}{(\alpha+k)!}\,.$$

This can be shown indeed by induction with respect to n; for $n=0$ see (5.11), (5.12). Hence

$$(11.6)\qquad \frac{Q^{(\alpha)}_{n+\mu+\nu}(x)}{\alpha!} = \sum_{k=0}^{\nu}\binom{\nu}{k}(-x)^k\frac{Q^{(\alpha+k)}_{n+\mu}(x)}{(\alpha+k)!}\,,$$

$$(11.7)\qquad \frac{Q^{(\alpha+\nu)}_{n+\mu}(x)}{(\alpha+\nu)!} = \sum_{k=0}^{\mu}\binom{\mu}{k}(-x)^k\frac{Q^{(\alpha+\nu+k)}_n(x)}{(\alpha+\nu+k)!}\,.$$

In the language of the matrix algebra, (11.6) and (11.7) can be written in the following concise form (μ, ν, k run from 0 to $l-1$):

$$\left(\frac{Q^{(\alpha)}_{n+\mu+\nu}(x)}{\alpha!}\right) = \left(\frac{Q^{(\alpha+\nu)}_{n+\mu}(x)}{(\alpha+\nu)!}\right)\cdot\left(\binom{\nu}{\mu}(-x)^{\mu}\right),$$

$$\left(\frac{Q^{(\alpha+\nu)}_{n+\mu}(x)}{(\alpha+\nu)!}\right) = \left(\binom{\mu}{\nu}(-x)^{\nu}\right)\cdot\left(\frac{Q^{(\alpha+\nu+\mu)}_n(x)}{(\alpha+\nu+\mu)!}\right).$$

Passing to the determinants we obtain

$$\left|\frac{Q^{(\alpha)}_{n+\mu+\nu}(x)}{\alpha!}\right|_0^{l-1} = \left|\frac{Q^{(\alpha+\nu)}_{n+\mu}(x)}{(\alpha+\nu)!}\right|_0^{l-1}\cdot\prod_{\mu=0}^{l-1}(-x)^{\mu},$$

$$\left|\frac{Q^{(\alpha+\nu)}_{n+\mu}(x)}{(\alpha+\nu)!}\right|_0^{l-1} = \prod_{\nu=0}^{l-1}(-x)^{\nu}\cdot\left[\frac{Q^{(\alpha+\nu+\mu)}_n(x)}{(\alpha+\nu+\mu)!}\right]_0^{l-1}$$

so that

$$(11.8)\qquad [Q^{(\alpha)}_{n+\mu+\nu}(x)]_0^{l-1} = (\alpha!)^l\prod_{\nu=0}^{l-1}x^{2\nu}\cdot\left[\frac{Q^{(\alpha+\nu+\mu)}_n(x)}{(\alpha+\nu+\mu)!}\right]_0^{l-1}.$$

Remark. Needless to say that all identities of this section, in

particular (11.8), hold for arbitrary values of α ($\alpha!$ etc. must be replaced by $\Gamma(\alpha+1)$) although in their immediate use we shall assume that α is an integer.

3. We return to (11.4). With the aid of (11.5) we can write

$$\frac{Q_{n+l-1-\mu}^{(\alpha+\nu+\mu)}(x)}{(\alpha+\nu+\mu)!}=\sum_{k=0}^{l-1-\mu}\binom{l-1-\mu}{k}(-x)^k\frac{Q_n^{(\alpha+\nu+\mu+k)}(x)}{(\alpha+\nu+\mu+k)!}$$
$$=\sum_{k=0}^{l-1-\mu}\binom{l-1-\mu}{k}(-x)^{l-1-\mu-k}\frac{Q_n^{(\alpha+\nu+l-1-k)}(x)}{(\alpha+\nu+l-1-k)!};$$

the second sum arises from the first one by replacing k by $l-1-\mu-k$. In the second sum we may extend the summation over $0\leq k\leq l-1$. Hence we have the further matrix identity

$$\left(\frac{Q_{n+l-1-\mu}^{(\alpha+\nu+\mu)}(x)}{(\alpha+\nu+\mu)!}\right)=\left(\binom{l-1-\mu}{\nu}(-x)^{l-1-\mu-\nu}\right)\cdot\left(\frac{Q_n^{(\alpha+\nu+l-1-\mu)}(x)}{(\alpha+\nu+l-1-\mu)!}\right).$$

We pass to the determinants and replace in both determinants on the right μ by $l-1-\mu$. We obtain

$$\left[\frac{Q_{n+l-1-\mu}^{(\alpha+\nu+\mu)}(x)}{(\alpha+\nu+\mu)!}\right]_0^{l-1}=\left[\binom{\mu}{\nu}(-x)^{\mu-\nu}\right]_0^{l-1}\cdot\left[\frac{Q_n^{(\alpha+\nu+\mu)}(x)}{(\alpha+\nu+\mu)!}\right]_0^{l-1}$$
$$=\left[\frac{Q_n^{(\alpha+\nu+\mu)}(x)}{(\alpha+\nu+\mu)!}\right]_0^{l-1}.$$

Comparing this with (11.8) we find

$$\text{(11.9)}\qquad [Q_{n+\mu+\nu}^{(\alpha)}(x)]_0^{l-1}=(\alpha!)^l\left(\prod_{\nu=0}^{l-1}x^{2\nu}\right)\cdot\left[\frac{Q_{n+l-1-\mu}^{(\alpha+\nu+\mu)}(x)}{(\alpha+\nu+\mu)!}\right]_0^{l-1}$$

so that in view of (11.4) we obtain indeed the special case of Theorem 5 mentioned above, i.e. the case of the Laguerre polynomials $Q_n^{(\alpha)}(x)$ with integer α, $\alpha=0, 1, 2, \ldots$.

Chapter 3. DETERMINANTS OF THE TURAN TYPE

§12. Theorem 5. Legendre polynomials: An identity.

The complete proof of Theorem 5 will require lengthy considerations. First we shall deal with the special case of the Legendre polynomials in which case a comparatively simple argument leads to the required result.

Then we shall proceed to the more complicated general ultraspherical and Laguerre polynomials, and finally to the Hermite polynomials; the latter case will offer only minor difficulties.

In all these cases the argument will be based mainly on a certain transformation of the given determinant of the Turán type into one of the Wronski type. We shall prove then that the latter one never vanishes on a proper range, provided the order of the determinants is even. We use the symbols T and W as introduced in § 1.

We denote by $P_n(x)$ the polynomial of Legendre, by $T_n(x)$ that of Tchebychev both of degree n. In Lemma 1 the number l is an arbitrary integer, in Lemma 2 (and in Theorem 5 of course) l must be even.

1. We prove first

Lemma 1: Using the notation of § 1, we have for all integer l and n, $l \geqq 1$, $n \geqq 1$, the following identity:

(12.1) $$T(P_n(x),\, P_{n+1}(x), \dots, P_{n+l-1}(x))$$
$$= A_{ln}(x^2-1)^{\frac{l(l-1)}{2}+\frac{ln}{2}} \cdot W(T_l(u),\, T_{l+1}(u), \dots, T_{l+n-1}(u)),\quad u = -\frac{x}{\sqrt{x^2-1}}.$$

Here A_{ln} is a constant depending only on l and n; we have

(12.2) $$A_{ln} = \frac{(-1)^{ln}}{2}\prod_{p=0}^{l-1} 2^{1-n-2p} \cdot \prod_{q=0}^{n-1}\{2^{1-q}(q!)^{-1}\}.$$

We note that (12.1) holds also for $n=0$ provided the Wronskian W in (12.1) and the second product in (12.2) are suppressed. Thus we have the formula

(12.3) $$T(P_0(x),\, P_1(x), \dots, P_{l-1}(x)) = \prod_{p=1}^{l-1} 2^{1-2p} \cdot (x^2-1)^{\frac{l(l-1)}{2}}$$

which agrees with (4.9).

The special case $l=1$ is also remarkable. We have then the representation

(12.4) $$P_n(x) = (-1)^n\, 2^{-n}\prod_{q=0}^{n-1}\{2^{1-q}(q!)^{-1}\} \cdot (x^2-1)^{\frac{n}{2}}$$
$$\cdot\, W(T_1(u),\, T_2(u), \dots, T_n(u)).$$

Concerning $n=0$ see above.

The Wronskian $W = W_l$ occuring in (12.1) is formed with respect to the variable u. More generally we shall consider the Wronskian W_p of the polynomials $T_p(u), T_{p+1}(u), \ldots, T_{p+n-1}(u)$, $0 \leqq p \leqq l$; since these polynomials are linearly independent, W_p is a non-identically vanishing polynomial. According to (6.1) its leading term is for $u \to \infty$, apart from trivial positive factors, u^{pn}. (See the remark to (6.1); the highest coefficients of the Tchebychev polynomials are positive.) For $x \to \pm 1$ we have $u \to \infty$ so that in view of (12.1) and (12.2) we find the following (cf. § 1.8):

The determinant T on the left of (12.1) as a polynomial in x has a zero of order $\frac{l(l-1)}{2}$ at $x = +1$ and $x = -1$. Moreover $\operatorname{sgn} T = (-1)^{ln}$ for $x < -1$, $T > 0$ for $x > 1$.

2. Let $l \geqq 1$, $n \geqq 1$. For convenience we assume first that $x > 1$ so that $u < -1$. We proceed as in § 4.2 where $[A]$ is now the determinant T, $a_{\mu\nu} = P_{n+\mu+\nu}(x)$. We choose H according to the condition $h_{\mu p} = 0$ for $\mu > p$ moreover so that the polynomials in t (x is a parameter)

$$\sum_{\mu=0}^{p} h_{\mu p}(x + \sqrt{x^2 - 1}\, t)^{\mu} = h_p(t) \tag{12.5}$$

satisfy the relations

$$\frac{1}{\pi}\int_{-1}^{+1} h_p(t)\, h_q(t)\,(x + \sqrt{x^2 - 1}\, t)^n \frac{dt}{\sqrt{1-t^2}} = \delta_{pq}, \quad p, q = 0, 1, \ldots, l-1. \tag{12.6}$$

(We take $\sqrt{x^2 - 1}$ as positive). With other words, the polynomials $h_p(t)$ will form an orthonormal system relative to the weight function

$$\frac{(\sqrt{x^2-1})^n}{\pi}(t-u)^n \cdot \frac{1}{\sqrt{1-t^2}}, \quad -1 < t < 1, \quad u = -\frac{x}{\sqrt{x^2-1}} < -1. \tag{12.7}$$

Indeed, we have then, cf. (4.5),

$$\begin{aligned} b_{pq} &= \sum h_{\mu p}\, a_{\mu\nu}\, h_{\nu q} = \sum h_{\mu p}\, h_{\nu q}\, P_{n+\mu+\nu}(x) \\ &= \frac{1}{\pi}\int_{-1}^{+1} \sum h_{\mu p}\, h_{\nu q}(x + \sqrt{x^2-1}\, t)^{\mu+\nu} \cdot (x + \sqrt{x^2-1}\, t)^n \frac{dt}{\sqrt{1-t^2}} = \delta_{pq} \end{aligned} \tag{12.8}$$

so that $[B] = 1$.

3. The orthonormal polynomials just introduced can easily be determined by using a formula of Christoffel [see 13, pp. 29—31] We form the polynomials in t (u is a parameter, $T_p(u)$ has the same meaning as in Lemma 1)

$$(12.9)\qquad \begin{vmatrix} T_p(t) & T_{p+1}(t) & \dots & T_{p+n}(t) \\ T_p(u) & T_{p+1}(u) & \dots & T_{p+n}(u) \\ T'_p(u) & T'_{p+1}(u) & \dots & T'_{p+n}(u) \\ \cdot & \cdot & \dots & \cdot \\ T_p^{(n-1)}(u) & T_{p+1}^{(n-1)}(u) & \dots & T_{p+n}^{(n-1)}(u) \end{vmatrix} = (x+\sqrt{x^2-1}\,t)^n g_p(t)$$

where $g_p(t)$ is of the precise degree p (cf. below). Clearly, $g_p(t)$ is orthogonal to all polynomials of degree $p-1$ relative to the weight (12.7). Hence the polynomials $g_p(t)$ will yield, apart from appropriate constant factors, the required polynomials $h_p(t)$. (The constant factors may depend on the parameters x or u.)

We find

$$(12.10)\qquad g_p(t) = (-1)^n \cdot (\sqrt{x^2-1})^{-n} \cdot 2^{p+n-1} \cdot W_p(u) \cdot t^p + \dots$$

where $W_p(u)$ stands for the Wronskian defined in **1**. All determinants $W_p(u)$, $0 \leqq p \leqq l$, occuring in the present proof are of the same order n. They are different from zero from $u<0$ (cf. Theorem 1), and the leading power of $W_p(u)$ is u^{pn}, $\operatorname{sgn} W_p(u) = (-1)^{pn}$. Now

$$(12.11)\qquad \frac{1}{\pi}\int_{-1}^{+1} (x+\sqrt{x^2-1}\,t)^n [g_p(t)]^2 \frac{dt}{\sqrt{1-t^2}}$$

$$= (-1)^n \cdot (\sqrt{x^2-1})^{-n} \cdot 2^{p+n-1} \cdot W_p(u) \cdot \frac{1}{\pi}\int_{-1}^{+1} (x+\sqrt{x^2-1}\,t)^n g_p(t) \frac{t^p\,dt}{\sqrt{1-t^2}}$$

$$= (-1)^n \cdot (\sqrt{x^2-1})^{-n} \cdot 2^{p+n-1} \cdot W_p(u) W_{p+1}(u) \cdot \frac{1}{\pi}\int_{-1}^{+1} T_p(t) \frac{t^p\,dt}{\sqrt{1-t^2}}$$

$$= (-1)^n \cdot (\sqrt{x^2-1})^{-n} \cdot 2^{n-1} \cdot W_p(u) W_{p+1}(u) = \gamma_p > 0\,.$$

[The positivity of γ_p follows from the nature of the left hand integral, $x>1$; we see this also from $\operatorname{sgn} W_p(u) = (-1)^{pn}$].

Consequently,

$$(12.12)\qquad h_p(t) = \gamma_p^{-1/2}\, g_p(t)\,.$$

This yields for the coefficients h_{pp} of $h_p(t)$, cf. (12.5),

$$h_{pp}(\sqrt{x^2-1})^p = \gamma_p^{-1/2} \cdot (-1)^n \cdot (\sqrt{x^2-1})^{-n} \cdot 2^{p+n-1} \cdot W_p(u),$$

$$(12.13) \quad h_{pp}^{-2} = (x^2-1)^{p+n} \cdot (-1)^n (\sqrt{x^2-1})^{-n} \cdot 2^{n-1} \cdot W_p(u) W_{p+1}(u) \cdot 2^{-2(p+n-1)} [W_p(u)]^{-2}, \quad p = 0, 1, \ldots, l-1,$$

so that

$$(12.14) \quad [A] = [H]^{-2} = (x^2-1)^{\frac{l(l-1)}{2}+\frac{ln}{2}} \cdot (-1)^{ln} \prod_{p=0}^{l-1} \left\{ 2^{1-n-2p} \frac{W_{p+1}(u)}{W_p(u)} \right\}$$
$$= (x^2-1)^{\frac{l(l-1)}{2}+\frac{ln}{2}} \cdot (-1)^{ln} \prod_{p=0}^{l-1} 2^{1-n-2p} \cdot \frac{W_l(u)}{W_0(u)}.$$

Inserting the constant value

$$(12.15) \qquad W_0(u) = T_0(u) T'_1(u) \ldots T_{n-1}^{(n-1)}(u) = 2 \prod_{q=0}^{n-1} \{2^{q-1} q!\}$$

we obtain (12.1). We proved this formula under the assumption $x > 1$; it holds of course for all x and in particular for $-1 < x < 1$ in which case u is purely imaginary.

In the case $n = 0$ the following changes are necessary:

$$h_p(t) = \sqrt{2}\, g_p(t) = \sqrt{2}\, T_p(t), \quad \gamma_p = \frac{1}{2}, \quad h_{pp} = 2^{p-\frac{1}{2}}, \quad p \geqq 1;$$
$$h_0(t) = g_0(t) = \gamma_0 = h_{00} = 1.$$

§ 13. Theorem 1, Legendre polynomials: The Wronskian.

We turn to the discussion of sgn W where $W = W_l$ is the determinant of Wronski type occuring in Lemma 1; we assume now that l is even, $-1 < x < 1$, hence u purely imaginary. We assume also that $n \geqq 1$. The condition that the order l of the determinant T is even is essential; indeed, for instance, for $l = 1$ we have $T = P_n = P_n(x)$ and this function changes its sign n times in the interval $-1 < x < 1$.

1. In this section we prove

Lemma 2: Let l and n be positive integers, l even, $n \geqq 1$. We have then

(13.1) $(-1)^{ln/2}\, W(T_l(u), T_{l+1}(u), \ldots, T_{l+n-1}(u)) = (-1)^{ln/2}\, W_l(u) > 0$

provided u is purely imaginary ($u = 0$ included).

For $n = 1$ we have $W_l(u) = T_l(u)$ so that the assertion is clear. We assume that $n \geqq 2$. We may also assume that $l \geqq 2$ since the case $l = 0$ is trivial.

In the case l odd, $n = 1$, the polynomial $W_l(u)$ vanishes for $u = 0$. For $l = 1$, $n = 2$ we have $W_l(u) = 2u^2 + 1$, and for $l = 3$, $n = 2$, complex (not purely imaginary) zeros appear.

2. Let $u = \frac{1}{2}(v + v^{-1})$, v purely imaginary, $|v| \geqq 1$. We have for any integral value of h

$$(13.2) \quad \left(\frac{d}{dv}\right)^h = t_1 \frac{d}{du} + t_2 \left(\frac{d}{du}\right)^2 + \ldots + t_{h-1}\left(\frac{d}{du}\right)^{h-1} + \left(\frac{du}{dv}\right)^h \left(\frac{d}{du}\right)^h$$

where $t_1, t_2, \ldots, t_{h-1}$ are certain functions of u (or v) depending on h. This formula can be verified by induction. But denoting by $\Delta(u)$ the Wronskian of $T_l, T_{l+1}, \ldots, T_{l+n-1}$ with respect to v, we have

$$(13.3) \qquad \Delta(u) = \left(\frac{du}{dv}\right)^{1+2+\ldots+n-1} \cdot W_l(u).$$

Since $\dfrac{du}{dv} = \dfrac{1}{2}(1 - v^{-2})$ is real and positive, the problem reduces to the evaluation of the sign of $\Delta(u)$.

Now $T_l(u) = \dfrac{1}{2}(v^l + v^{-l})$ so that the first column of $\Delta(u)$ contains the following quantities:

$$\frac{1}{2}(v^l + v^{-l}), \quad \frac{1}{2} l v^{l-1} + \frac{1}{2}(-l)\, v^{-l-1}, \ldots,$$

$$\frac{1}{2} l(l-1) \ldots (l-n+2)\, v^{l-n+1} + \frac{1}{2}(-l)(-l-1) \ldots (-l-n+2)\, v^{-l-n+1}.$$

The later columns arise by replacing l by $l+1; l+2, \ldots, l+n-1$, respectively. Multiplying the rows by $1, v, v^2, \ldots, v^{n-1}$, respectively, we obtain for $2^n v^{n(n-1)/2}\, \Delta(u)$ the elements

$$(13.4) \qquad \begin{aligned} &v^l + v^{-l},\ l v^l + (-l)\, v^{-l}, \ldots, \\ &l(l-1) \ldots (l-n+2)\, v^l + (-l)(-l-1) \ldots (-l-n+2)\, v^{-l} \end{aligned}$$

in the first column, and correspondingly in the later columns.

3. Since l is even and v purely imaginary, the elements of the first column are real, those of the second purely imaginary, those of the third real, etc. Suppose now that $\Delta(u) = 0$ for a certain purely imaginary value of v, $|v| \geqq 1$. There exist then certain real constants $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ not all zero such that writing

$$(13.5) \quad \alpha_0 + \alpha_1 x + \alpha_2 x(x-1) + \ldots + \alpha_{n-1} x(x-1)\ldots(x-n+2) = \alpha(x),$$

we have

$$(13.6) \quad \alpha(l) v^l + \alpha(-l) v^{-l} = 0, \quad \alpha(l+1) v^{l+1} + \alpha(-l-1) v^{-l-1} = 0, \ldots,$$
$$\alpha(l+n-1) v^{l+n-1} + \alpha(-l-n+1) v^{-l-n+1} = 0.$$

Hence for $x = l, l+1, \ldots, l+n-1$ we find that

$$(13.7) \qquad \beta(x) = \alpha(x)\alpha(-x) = -[\alpha(-x)]^2 v^{-2x}$$

is of the sign $(-1)^{l-1}, (-1)^l, \ldots, (-1)^{l+n-2}$, or 0. (We recall that v is purely imaginary.) Since l is even, we have

$$(13.8) \qquad \beta(l) \leqq 0, \beta(l+1) \geqq 0, \ldots, (-1)^n \beta(l+n-1) \geqq 0.$$

We note also that $\beta(0) = \alpha_0^2 \geqq 0$, moreover that $\beta(x)$ is an even polynomial of degree $2n-2$.

Thus we conclude the existence of certain values $x_1, x_2, \ldots, x_n$ such that

$$(13.9) \qquad 0 < x_1 < l < x_2 < l+1 < \ldots < l+n-2 < x_n < l+n-1$$

and

$$(13.10) \qquad \beta'(x_1) < 0, \ \beta'(x_2) > 0, \ldots, (-1)^n \beta'(x_n) > 0.$$

Consequently the odd polynomial $\beta'(x)$ of degree $2n-3$ vanishes for certain $n-1$ distinct positive values of x, and of course also for the corresponding negative values. The total number of these zeros being $2n-2$, $\beta'(x)$ must be identically zero which is a contradiction to (13.10).

4. In order to evaluate the constant value $\operatorname{sgn} W_l(u)$, u purely imaginary, we use the remark in § 12.1 according to which the leading term of $W_l(u)$ is u^{ln}. Since u is purely imaginary and l even, the sign in question is indeed $(-1)^{ln/2}$.

Finally the assertion of Theorem 5 (for Legendre polynomials) follows readily by combining Lemma's 1 and 2; if l is even, we have indeed by

(12.1), (12,2), (13,1)

$$\operatorname{sgn} T = (-1)^{\frac{l(l-1)}{2}+\frac{ln}{2}} \cdot (-1)^{\frac{ln}{2}} = (-1)^{\frac{l}{2}}.$$

§ 14. Theorem 5, ultraspherical polynomials: An identity.

1. Our further aim is to prove the following generalization of Lemma 1.

Lemma 3: The following identity holds for all integer values of l and n, $l \geqq 1$, $n \geqq 1$:

$$T\left(\frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)}, \frac{P_{n+1}^{(\lambda)}(x)}{P_{n+1}^{(\lambda)}(1)}, \ldots, \frac{P_{n+l-1}^{(\lambda)}(x)}{P_{n+l-1}^{(\lambda)}(1)}\right)$$

$$(14.1) \qquad = A_{ln}^{(\lambda)} (x^2-1)^{\frac{l(l-1)}{2}+\frac{ln}{2}} \cdot W\left(P_l^{(\mu)}(u), P_{l+1}^{(\mu)}(u), \ldots, P_{l+n-1}^{(\mu)}(u)\right).$$

$$\mu = \lambda - \frac{1}{2}, \quad u = \frac{x}{\sqrt{x^2-1}},$$

where the constant $A_{ln}^{(\lambda)}$ depends only on l, n, λ; we have

$$(14.2) \qquad A_{ln}^{(\lambda)} = (-1)^{ln} \prod_{p=0}^{l-1} \{(\rho_p \rho_{p+n})^{-1} \sigma_p\} \cdot \prod_{q=0}^{n-1} (\rho_q\, q!)^{-1}.$$

The quantities ρ_p, σ_p are defined by (14.5).

The parameters λ and μ will be always connected by the relation $\mu = \lambda - \frac{1}{2}$. The case $-1 < x < 1$ is again of main importance; then u is purely imaginary. Lemma 1 arises for $\lambda = \frac{1}{2}$, $\mu = 0$ which is actually a limiting case, cf. (14.8). The Wronskian W is formed of course by differentiating with respect to the variable u. The sign of $A_{ln}^{(\lambda)}$ will be discussed at the end of this section.

For $n = 0$ the Wronskian in (14.1) and the second product in (14.2) must be suppressed and we have the formula

$$(14.3) \quad T\left(\frac{P_0^{(\lambda)}(x)}{P_0^{(\lambda)}(1)}, \frac{P_1^{(\lambda)}(x)}{P_1^{(\lambda)}(1)}, \ldots, \frac{P_{l-1}^{(\lambda)}(x)}{P_{l-1}^{(\lambda)}(1)}\right) = \prod_{p=0}^{l-1} \{\rho_p^{-2} \sigma_l\} \cdot (x^2-1)^{\frac{l(l-1)}{2}}$$

In the special case $l = 1$ we have

$$(14.4) \qquad \frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)} = (-1)^n \rho_n^{-1} \prod_{q=0}^{n-1} (\rho_q\, q!)^{-1} \cdot (x^2-1)^{\frac{n}{2}}$$

$$\cdot\, W\left(P_1^{(\mu)}(u), P_2^{(\mu)}(u), \ldots, P_n^{(\mu)}(u)\right).$$

The cases $\lambda = \frac{1}{2}$ and $\lambda = 0$ are "singular"; it is advisable to treat them later as limiting cases.

As in § 12, the determinant T on the left of (14.1) has a zero of order $\frac{l(l-1)}{2}$ at $x = \pm 1$ and $\operatorname{sgn} T = (-1)^{ln}$ for $x < -1$, $T > 0$ for $x > 1$ provided that $\lambda > 0$. If $-\frac{1}{2} < \lambda < 0$ the same holds for $(-1)^l T$.

2. In what follows we shall make use of the familiar properties of the ultraspherical polynomials referring in particular to the formulas (5.3)—(5.10). We shall use the notation adopted there. We introduce the following symbols:

$$(14.5) \qquad \rho_n = k_n^{(\lambda-\frac{1}{2})} = 2^n \binom{n+\lambda-\frac{3}{2}}{n}, \quad \sigma_n = \pi^{-\frac{1}{2}} \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)} h_n^{(\lambda-\frac{1}{2})}$$

$$= \frac{\lambda - \frac{1}{2}}{n + \lambda - \frac{1}{2}} \binom{n + 2\lambda - 2}{n}.$$

We have $\rho_0 = \sigma_0 = 1$.

3. Since (14.1) is an identity not only in x but also in λ [$P_n^{(\lambda)}(x)$ is a polynomial in λ, cf. (5.3)] we may assume for the proof that $\lambda > \frac{1}{2}$, i.e. $\mu > 0$, and $x > 1$, i.e. $u > -1$; also let $\sqrt{x^2-1} > 0$. Following closely the argument of § 12 we can be brief. The polynomials

$$h_p(t) = h_{pp}(x + \sqrt{x^2-1}\, t)^p + \ldots$$

of (12.5) must be generalized now as to satisfy the orthogonality relations

$$(14.6) \qquad \pi^{-\frac{1}{2}} \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)} \int_{-1}^{+1} h_p(t)\, h_q(t)\, (x+\sqrt{x^2-1}\, t)^n (1-t^2)^{\lambda-1}\, dt = \delta_{pq},$$

$$p, q = 0, 1, \ldots, l-1.$$

[We note that $(1-t^2)^{\lambda-1} = (1-t^2)^{\mu-\frac{1}{2}}$ so that for $n = 0$ we have $h_p(t) = \text{const.}\ P_p^{(\mu)}(t)$]. We form therefore, in generalizing (12.9), x and u are parameters,

$$(14.7)\quad \begin{vmatrix} f_p(t) & f_{p+1}(t) & \dots & f_{p+n}(t) \\ f_p(u) & f_{p+1}(u) & \dots & f_{p+n}(u) \\ \cdot & \cdot & \dots & \cdot \\ f_p^{(n-1)}(u) & f_{p+1}^{(n-1)}(u) & \dots & f_{p+n}^{(n-1)}(u) \end{vmatrix} = (x+\sqrt{x^2-1}\,t)^n\, g_p(t)\,;$$

$$f_p(t) = P_p^{(\mu)}(t) = \rho_p t^p + \dots$$

where ρ_p, and later σ_p, have the meaning (14.5). We generalize (12.10) and (12.11) as follows:

$$g_p(t) = (-1)^n \cdot (\sqrt{x^2-1})^{-n} \cdot \rho_{p+n} \cdot W_p(u) \cdot t^p + \dots,$$

$$\pi^{-\frac{1}{2}} \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)} \int_{-1}^{+1} (x+\sqrt{x^2-1}\,t)^n\,[g_p(t)]^2 (1-t^2)^{\lambda-1}\,dt$$

$$= (-1)^n \cdot (\sqrt{x^2-1})^{-n} \cdot \rho_{p+n} \cdot W_p(u)\, W_{p+1}(u)$$

$$\cdot\ \pi^{-\frac{1}{2}} \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)} \int_{-1}^{+1} f_p(t)\, t^p (1-t^2)^{\lambda-1}\,dt$$

$$= (-1)^n \cdot (\sqrt{x^2-1})^{-n} \cdot \frac{\rho_{p+n}}{\rho_p} \cdot W_p(u)\, W_{p+1}(u) \cdot \sigma_p = \gamma_p$$

where $W_p(u)$ stands for the Wronskian $W(P_p^{(\mu)}(u), P_{p+1}^{(\mu)}(u), \dots, P_{p+n-1}^{(\mu)}(u))$; $p = 0, 1, \dots, l$. We made use of (5.7). The left hand integral makes it clear that γ_p is positive. (This follows also by observing that the leading term of $W_p(u)$ is, apart from trivial positive factors, u^{pn}, see (6.1), hence the sign of $W_p(u)$ is $(-1)^{pn}$). Thus (12.12) holds and

$$h_{pp}(\sqrt{x^2-1})^p = \gamma_p^{-\frac{1}{2}} \cdot (-1)^n \cdot (\sqrt{x^2-1})^{-n} \cdot \rho_{p+n} \cdot W_p(u),$$

$$h_{pp}^{-2} = (x^2-1)^{p+n} \cdot (-1)^n \cdot (\sqrt{x^2-1})^{-n} \cdot (\rho_p \rho_{p+n})^{-1} \sigma_p$$
$$\cdot\ W_p(u)\, W_{p+1}(u) \cdot [W_p(u)]^{-2},$$
$$p = 0, 1, \dots, l-1\,.$$

Consequently,

$$[A] = T = (-1)^{ln} (x^2-1)^{\frac{l(l-1)}{2} + \frac{ln}{2}} \cdot \prod_{p=0}^{l-1} \{(\rho_p \rho_{p+n})^{-1} \sigma_p\} \cdot \frac{W_l(u)}{W_0(u)}\,.$$

In view of $W_0(u) = \prod_{q=0}^{n-1} (\rho_q\, q!)$ we obtain (14.1), (14.2).

The modifications necessary for $n = 0$ are obvious.

4. A brief analysis of sgn $A_{ln}^{(\lambda)}$ will be useful.

Let $\lambda > -\frac{1}{2}$ and l arbitrary (even or odd). The factors of $(-1)^{ln} A_{ln}^{(\lambda)}$ are all positive when $\lambda > \frac{1}{2}$. The only zeros or poles appear for $\lambda = \frac{1}{2}$ (Legendre polynomials) and $\lambda = 0$ (Tchebychev polynomials). We find in $\rho_p, p \geqq 1$, the factor $\lambda - \frac{1}{2}$ and in $\sigma_p, p \geqq 2$, the factor $\lambda(\lambda - \frac{1}{2})^2$; moreover $\rho_0 = \sigma_0 = 1$ and σ_1 has the only factor $(\lambda - \frac{1}{2})^2$. Thus the only "essential" factors of $(-1)^{ln} A_{ln}^{(\lambda)}$ are $(\lambda - \frac{1}{2})^{-n}$. λ^{l-2} where the second factor is non-existent if $l \leqq 2$.

Now we factor out from $W = W_l(u)$ the highest coefficients $\rho_l, \rho_{l+1}, \dots, \rho_{l+n-1}$ of the ultraspherical polynomials occuring in $W_l(u)$ and form the product

$$(-1)^{ln} A_{ln}^{(\lambda)} \cdot \rho_l \rho_{l+1} \dots \rho_{l+n-1};$$

this expression will have the only essential factor λ^{l-2} and the quantity $(\rho_l \rho_{l+1} \dots \rho_{l+n-1})^{-1} \cdot W_l(u)$ (to be denoted later by $\overline{W}(\lambda, u)$, cf. (15.3)) will be regular and not identically zero for $\lambda > -\frac{1}{2}$. Hence the determinant $[A] = T$ vanishes identically in the Tchebychev case $\lambda \to 0$, $l \geqq 3$, and only in this case.

If $\lambda \to \frac{1}{2}$, $\mu \to 0$ we obtain, as a limiting case, Tchebychev's polynomials (see the footnote to (5.3)):

$$\text{(14.8)} \qquad \rho_n \cong \frac{2^n}{n} \mu, \quad \sigma_n \cong \frac{2}{n^2} \mu^2, \quad P_n^{(\mu)}(x) \cong \frac{2T_n(x)}{n} \mu; \quad n \geqq 1.$$

Then the Wronskian occuring in Lemma 1 follows.

If $\lambda \to 0$, $\mu \to -\frac{1}{2}$ we have

$$\text{(14.9)} \qquad \rho_n = 2^n \binom{n - \frac{3}{2}}{n}; \quad \sigma_n \cong \frac{\lambda}{n(n-1)(n-\frac{1}{2})}, \quad n \geqq 2; \ \sigma_1 \to 1,$$

and the product $\lambda^{-(l-2)} T$ tends to a limit which is (apart from familiar factors) the Wronskian of the functions $P_p^{-(1/2)}(u)$ [cf. 13, p. 384, Problem 62].

§ 15. Theorem 5, ultraspherical polynomials, l even: The Wronskian.

1. We prove now the following

Lemma 4: Let l and n be positive integers, l even, $n \geqq 1$. We have then

$$(15.1)\quad (-1)^{ln/2}\, W\left(P_l^{(\mu)}(u), P_{l+1}^{(\mu)}(u), \dots, P_{l+n-1}^{(\mu)}(u)\right) = (-1)^{ln/2}\, W_l(u) > 0,$$
$$\mu = \lambda - \tfrac{1}{2},$$

provided u is purely imaginary ($u = 0$ included) and $\lambda > \frac{1}{2}$. For $-\frac{1}{2} < \lambda < \frac{1}{2}$ the constant sign of $W_l(u)$ is $(-1)^{ln/2+n}$. For $\lambda = \frac{1}{2}$, $\mu = 0$ the Wronskian vanishes identically but taking out the factor $(\lambda - \frac{1}{2})^n = \mu^n$ it reduces (apart from trivial positive factors) to the Wronskian considered in Lemma 2 and it has the constant sign $(-1)^{ln/2}$.

The cases $n = 1$ or $l = 0$ are trivial. We assume that $n \geqq 2$, $l \geqq 2$.

The proof is based (a) on induction with respect to n (for $n = 1$ the assertion is obvious), and (b) on the special case $\lambda = \frac{1}{2}$, $\mu = 0$ which was settled in Lemma 2. The method is completely different from that used in the case $\lambda = \frac{1}{2}$.

2. Using (5.3) we find

$$(15.2)\quad \rho_p^{-1}\, P_p^{(\mu)}(u) = \sum_{\nu=0}^{[p/2]} (-1)^\nu \frac{\Gamma(p-\nu+\lambda-\frac{1}{2})}{\Gamma(p+\lambda-\frac{1}{2})} \frac{\Gamma(p+1)}{\Gamma(\nu+1)\,\Gamma(p-2\nu+1)}\, 2^{-2\nu} u^{p-2\nu}$$
$$= u^p + \dots .$$

Now we form

$$(15.3)\qquad (\rho_l \rho_{l+1} \dots \rho_{l+n-1})^{-1} \cdot W_l(u) = \overline{W}(\lambda, u).$$

This is a polynomial in u and a rational function of λ regular for $\lambda > -\frac{1}{2}$ (incl. $\lambda = \infty$). We can indicate its structure as follows:

$$\overline{W}(\lambda, u) = W\{u^l + (\lambda, u)_l,\ u^{l+1} + (\lambda, u)_{l+1}, \dots, u^{l+n-1} + (\lambda, u)_{l+n-1}\}$$

where $(\lambda, u)_p$ designates a polynomial in u containing only the powers $u^{p-2}, u^{p-4}, \dots$; $(\lambda, u)_p$ is a rational function of λ regular for $\lambda > -\frac{1}{2}$ and vanishing for $\lambda \to \infty$. The coefficients are all real.

Multiplying the rows of $\overline{W}$ by $1, u, u^2, \dots, u^{n-1}$ and then the columns by $u^{-l}, u^{-l-1}, \dots, u^{-l-n+1}$ we obtain $u^{-ln}\, W$; its leading term about $u = \infty$ will be the Vandermondian of the constants $l, l+1, \dots, l+n-1$ (cf. § 6.1). Thus $u^{-ln}\, \overline{W}$ is a rational function of λ, regular for $\lambda > -\frac{1}{2}$ (incl. $\lambda = \infty$), and a polynomial in $\frac{1}{u}$; it will reduce to a positive constant independent of λ as $u \to \infty$. Consequently, ε being a given positive number,

we can find a positive constant $v_0 = v_0(l, n, \varepsilon)$ such that $(-1)^{ln/2}\ \overline{W} > 0$ for $u = iv$, v real, $|v| \geqq v_0$ for all values of $\lambda, \lambda \geqq -\frac{1}{2} + \varepsilon$.

3. For an arbitrary λ we consider now $\min(-1)^{ln/2}\ \overline{W} = m(\lambda)$ in the finite real interval $|v| \leqq v_0$; we have $m(\lambda) > 0$ for $\lambda = \frac{1}{2}$ (this is the special case considered in Lemma 2).[5] Since $m(\lambda)$ is a continuous function of λ, we have just two possibilities: either $m(\lambda) > 0$ for all $\lambda \geqq -\frac{1}{2} + \varepsilon$, or there is a value $\lambda = \lambda'$, $\lambda' \geqq -\frac{1}{2} + \varepsilon$, for which $m(\lambda') = 0$. Let $u = u'$ be the value of u (or one of the values) for which $m(\lambda') = 0$ is attained; $|u'| < v_0$. Then we have necessarily for $\lambda = \lambda'$, $u = u'$

$$\overline{W} = 0, \quad \frac{\partial \overline{W}}{\partial u} = 0. \tag{15.4}$$

4. We prove that (15.4) leads to a contradiction. We make use of the differential equation (5.5); denoting by $L(y)$ the operator

$$L(y) = (1 - u^2)y'' - (2\mu + 1)uy' \tag{15.5}$$

we have for $y = \rho_p^{-1} P_p^{(\mu)}(u)$ the identity $Ly = \gamma y$, $\gamma = \gamma_p = -p(p + 2\mu)$.

The general (p^{th}) column of $\overline{W}$ can be written as

$$y, y', y'', \ldots, y^{(n-1)}; \quad y = \rho_p^{-1} P_p^{(\mu)}(u), \quad p = l, l+1, \ldots, l+n-1. \tag{15.6}$$

Let n be even; replacing the elements (15.6) by

$$y, y', Ly, (Ly)', L^2y, (L^2y)', \ldots, L^{n/2-1}y, (L^{n/2-1}y)' \tag{15.7}$$

we obtain $(1-u^2)^m\ \overline{W}$, $m = \frac{n}{2}\left(\frac{n}{2} - 1\right)$; but (15.7) is identical with

$$\begin{gathered} y, y', \gamma y, \gamma y', \gamma^2 y, \gamma^2 y', \ldots, \gamma^{n/2-1} y, \gamma^{n/2-1} y'; \\ \gamma = \gamma_p, p = l, l+1, \ldots, l+n-1. \end{gathered} \tag{15.8}$$

We note that omitting the last row in (15.8) and keeping the first $n-1$ columns, $p = l, l+1, \ldots, l+n-2$, apart from a certain power of $(1-u^2)$ the Wronskian of the polynomials $P_l^{(\mu)}(u), \ldots, P_{l+n-2}^{(\mu)}(u)$ arises which is different from zero, by the hypothesis of the induction.

---

5. It does not seem possible to start the subsequent "continuity argument" with $\lambda = \infty$; indeed, for $\lambda \to \infty$ the functions (15.2) will become $u^l, u^{l+1}, \ldots, u^{l+n-1}$ and the Wronskian of the latter functions vanishes for $u = 0$.

We make now use of the hypothesis (15.4) by differentiating the single rows of the determinant whose p^{th} column (15.8) is, $\gamma = \gamma_p$; $p = l, l+1, \dots, l+n-1$. Differentiation of the rows in which the elements $y, \gamma y, \gamma^2 y, \dots$ occur, leads immediately to 0. Now let us differentiate the second row represented by y'. Combining this differentiated determinant with the original one and taking $(1-u^2)y'' - (2\mu+1)uy' = \gamma y$ into account, we see that the resulting determinant is again zero. The only exception is the last row; differentiating the last element in (15.8) and combining this properly with (15.8) we obtain $\gamma^{n/2} y$. Thus the determinant for which a typical column has the elements

$$(15.9) \qquad \begin{gathered} y, y', \gamma y, \gamma y', \gamma^2 y, \gamma^2 y', \dots, \gamma^{n/2-1} y, \gamma^{n/2} y; \\ y = \rho_p^{-1} P_p^{(\mu)}(u), \quad \gamma = \gamma_p, \end{gathered}$$

must also vanish.

We multiply every even numbered row of (15.8) and (15.9), except the last row of (15.9), by i. Then all elements of (15.8) and (15.9) become real if p is even and purely imaginary if p is odd. We conclude the existence of four polynomials

$$(15.10) \qquad \begin{cases} f_0(t) = a_0 + a_1 t + \dots + a_{n/2-1} t^{n/2-1}, \\ f_1(t) = b_0 + b_1 t + \dots + b_{n/2-1} t^{n/2-1}, \\ f_2(t) = c_0 + c_1 t + \dots + c_{n/2} \quad t^{n/2}, \\ f_3(t) = d_0 + d_1 t + \dots + d_{n/2-2} t^{n/2-2}, \end{cases}$$

with r e a l coefficients such that the following equations hold:

$$(15.11) \qquad \begin{gathered} f_0(\gamma) y + f_1(\gamma) iy' = 0, \quad f_2(\gamma) y + f_3(\gamma) iy' = 0; \\ y = \rho_p^{-1} P_p^{(\mu)}(u), \quad \gamma = \gamma_p, \qquad p = l, l+1, \dots, l+n-1. \end{gathered}$$

We note that $b_{n/2-1} \neq 0$ and $c_{n/2} \neq 0$ (otherwise we arrive at a contradiction to the induction hypothesis (a)). Since y and iy' are not zero for $u \neq 0$ and at any rate not both zero for $u = 0$, we have

$$(15.12) \qquad f_0(\gamma) f_3(\gamma) - f_1(\gamma) f_2(\gamma) = 0.$$

This is an algebraic equation of the precise degree $\frac{n}{2} - 1 + \frac{n}{2} = n - 1$ which is satisfied for n distinct values of $\gamma = \gamma_l, \gamma_{l+1}, \dots, \gamma_{l+n-1}$; this is impossible.

L e t n b e o d d. Then (15.7) and (15.8) must be modified as follows:

$$y, y', Ly, (Ly)', \ldots, L^{(n-3)/2}y, (L^{(n-3)/2}y)', L^{(n-1)/2}y,$$
$$y, y', \gamma y, \gamma y', \ldots, \gamma^{(n-3)/2}y, \gamma^{(n-3)/2}y', \gamma^{(n-1)/2}y.$$

Instead of (15.9) we obtain now

$$y, y', \gamma y, \gamma y', \ldots, \gamma^{(n-3)/2}y, \gamma^{(n-3)/2}y', \gamma^{(n-1)/2}y',$$

so that (15.11) holds again; but this time the degrees of the polynomials occuring in (15.10) are

$$\frac{n-1}{2}, \frac{n-3}{2}, \frac{n-3}{2}, \frac{n-1}{2},$$

respectively, and the degrees of $f_0(t)$ and $f_3(t)$ are precisely $(n-1)/2$. Since (15.12) holds again, the contradiction follows as before.

Thus $\overline{W}$ keeps a constant sign for purely imaginary u which is $(-1)^{ln/2}$ since its leading term is u^{ln}.

This establishes the proof of Lemma 4. Combining the result with Lemma 3, in particular with (14.1), (14.2), and taking the remarks of § 14.4 into account, Theorem 5 for ultraspherical polynomials follows without difficulty.

§ 16. Theorem 5. Laguerre polynomials: An identity.

In § 11 we have proved Theorem 5 for the Laguerre polynomials $L_n^{(\alpha)}(x)/L_n^{(\alpha)}(0) = Q_n^{(\alpha)}(x)$ under the restriction that the parameter α has an *integer* value. In particular we know that Theorem 5 holds for the case of the Laguerre polynomials $L_n^{(0)}(x) = L_n(x)$. Based on this fact, we shall prove in § 17 that Theorem 5 holds for *all* $L_n^{(\alpha)}(x)/L_n^{(\alpha)}(0) = Q_n^{(\alpha)}(x)$, α arbitrary, $\alpha > -1$.

1. In the present section we prove as a preparation the following

Lemma 5: *Let l and n be integers, $l \geq 1$, $n \geq 1$. We have the identity*

$$T\left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}, \frac{L_{n+1}^{(\alpha)}(x)}{L_{n+1}^{(\alpha)}(0)}, \ldots, \frac{L_{n+l-1}^{(\alpha)}(x)}{L_{n+l-1}^{(\alpha)}(0)}\right) \tag{16.1}$$
$$= A_{ln}^{(\alpha)} x^{l(l-1)+ln} \cdot W(\lambda_l(u), \lambda_{l+1}(u), \ldots, \lambda_{l+n-1}(u)), \quad u = -\frac{1}{x},$$

where the constant $A_{ln}^{(\alpha)}$ depends only on l, n, α; moreover

(16.2) $$\lambda_p(u) = (-u)^p L_p^{(-\alpha-2p)}(u^{-1}).$$

We have

(16.3) $$A_{ln}^{(\alpha)} = (-1)^{\frac{l(l-1)}{2}+ln} \prod_{p=0}^{l-1} \left[\left\{ \binom{\alpha+2p+2n-1}{p+n} \right\}^{-1} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2p+1)} \right] \cdot \prod_{q=0}^{n-1} \left\{ \binom{\alpha+2q-1}{q} q! \right\}^{-1}.$$

The Wronskian is formed with respect to the variable u which will be always connected with x by the relation $u = -\frac{1}{x}$. We note that

(16.4) $$\operatorname{sgn} A_{ln}^{(\alpha)} = (-1)^{\frac{l(l-1)}{2}+ln}.$$

In the case $n = 0$ the formula is still valid with a slight modification: W and the second product in (16.3) have to be omitted so that the following simple analog of (12.3) and (14.3) arises:

(16.5) $$T\left(\frac{L_0^{(\alpha)}(x)}{L_0^{(\alpha)}(0)}, \frac{L_1^{(\alpha)}(x)}{L_1^{(\alpha)}(0)}, \ldots, \frac{L_{l-1}^{(\alpha)}(x)}{L_{l-1}^{(\alpha)}(0)} \right) = A \cdot x^{l(l-1)};$$

$$A = (-1)^{\frac{l(l-1)}{2}} \frac{1!\,2!\ldots(l-1)!\,[\Gamma(\alpha+1)]^l}{\Gamma(\alpha+l)\,\Gamma(\alpha+l+1)\ldots\Gamma(\alpha+2l-1)}.$$

In the other remarkable case $l = 1$ we have [cf. (12.4), (14.4)]:

(16.6) $$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} = (-1)^n n! \prod_{q=0}^{n} \left\{ \binom{\alpha+2q-1}{q} q! \right\}^{-1} \cdot x^n \cdot W(\lambda_1(u), \lambda_2(u), \ldots, \lambda_n(u)), \qquad u = -\frac{1}{x}.$$

2. In order to prove the identity (16.1) we may assume that x is negative so that $u > 0$. We may also assume that x is "sufficiently" small in a sense to be defined below. Finally let $\alpha > -1$. We see easily that [cf. (5.11)]

(16.7) $$\lambda_p(u) = \sum_{\nu=0}^{p} \binom{-\alpha-p}{p-\nu} \frac{(-u)^{p-\nu}}{\nu!} = \sum_{\nu=0}^{p} \binom{\alpha+p+\nu-1}{\nu} \frac{u^\nu}{(p-\nu)!} = \frac{1}{p!} + \ldots + \binom{\alpha+2p-1}{p} u^p.$$

The following differential equation holds:

$$u^2 \lambda''_p(u) + [1+(\alpha+1)u]\,\lambda'_p(u) - p(\alpha+p)\,\lambda_p(u) = 0. \tag{16.8}$$

This can be ascertained either from the differential equation of the Laguerre polynomials [(5.13)] or directly:

$$\sum_{\nu=0}^{p} \binom{\alpha+p+\nu-1}{\nu} \frac{1}{(p-\nu)!} \{\nu(\nu-1)u^\nu + [1+(\alpha+1)u]\,\nu u^{\nu-1} - p(\alpha+p)u^\nu\}$$
$$= \sum_{\nu=0}^{p} \binom{\alpha+p+\nu-1}{\nu} \frac{1}{(p-\nu)!} \{\nu(\nu-1) + (\alpha+1)\nu - p(\alpha+p)\}\,u^\nu$$
$$+ \sum_{\nu=1}^{p} \binom{\alpha+p+\nu-1}{\nu} \frac{1}{(p-\nu)!}\,\nu\,u^{\nu-1} = 0.$$

Equating the coefficients of u^ν the identity follows.

The leading term of the Wronskian

$$W(\lambda_p(u), \lambda_{p+1}(u), \ldots, \lambda_{p+n-1}(u)) = W_p(u)$$

is (apart from trivial positive factors) u^{pn} so that $W_p(u)$ is positive provided u is sufficiently large (or x is sufficiently small negative); $p = 0, 1, \ldots, l$. From now on we assume in this section that $W_p(u) > 0$, $0 \leqq p \leqq l$; this is the condition regarding x mentioned above.

In view of the remark about the leading term we conclude easily that the determinant T on the left of (16.1), as a polynomial in x, has a zero of order $l(l-1)$ at $x = 0$.

As a further preparation we note that, cf. (5.16),

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} = Q_n^{(\alpha)}(x) = \sum_{\rho=0}^{n} \binom{n}{\rho} k_\rho(x) = \sum_{\rho=0}^{\infty} \binom{n}{\rho} k_\rho(x), \tag{16.9}$$

$$k_\rho(x) = \frac{(-x)^\rho}{(\alpha+1)\ldots(\alpha+\rho)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\rho+1)}(-x)^\rho\ ;\ k_0(x) = 1. \tag{16.10}$$

Moreover we have the trivial formula

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{(1+z)^n}{z^\rho} \frac{dz}{z} = \binom{n}{\rho} \tag{16.11}$$

where n and ρ are non negative integers. The integration is extended over the unit circle in the positive direction.

It is convenient to define the following operator $K\{f(z)\}$ applicable

to any analytic function $f(z)$ regular for $|z| \leqq 1$:

$$(16.12) \qquad K\{f(z)\} = \frac{1}{2\pi i} \int\limits_{|z|=1} \sum_{\rho=0}^{\infty} \frac{k_\rho(x)}{z^\rho} \cdot f(z) \cdot \frac{dz}{z}.$$

(The operator K depends on the parameters α and x). We cite the following example:

$$(16.13) \qquad K\{z^\rho\} = k_\rho(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\rho+1)} (-x)^\rho.$$

3. Following the procedure of § 4.2, § 12 and § 14, we base the evaluation (transformation) of the determinant $T = [A]$ of (16.1) on the construction of certain polynomials

$$(16.14) \qquad h_p(z) = \sum_{\mu \leqq p} h_{\mu p} (1+z)^\mu$$

satisfying now the following "orthogonality relations" (6)

$$(16.15) \qquad \sum_{\mu, \nu = 0, 1, \ldots, l-1} h_{\mu p} Q_{n+\mu+\nu}^{(\alpha)}(x) h_{\nu q} = \sum_{\rho=0}^{\infty} k_\rho(x) \sum_{\mu, \nu} h_{\mu p} \binom{n+\mu+\nu}{\rho} h_{\nu q}$$

$$= \sum_{\rho=0}^{\infty} k_\rho(x) \sum_{\mu, \nu} h_{\mu p} \cdot \frac{1}{2\pi i} \int\limits_{|z|=1} \frac{(1+z)^{n+\mu+\nu}}{z^\rho} \frac{dz}{z} \cdot h_{\nu q}$$

$$= K\{(1+z)^n h_p(z) h_q(z)\} = \begin{cases} 0 & \text{if } p \neq q, \\ (-1)^{p+n} W_p(u) W_{p+1}(u) x^{-n} & \text{if } p = q, \end{cases}$$

$$p, q = 0, 1, \ldots, l-1.$$

The last expression (normalization) has been chosen conveniently as we shall see later; it is of the sign $(-1)^p$ since $x < 0$, $u = -\dfrac{1}{x} > 0$ and x has been chosen so small that all $W_p(u)$ are positive, $0 \leqq p \leqq l$; see **2.**

If such polynomials exist with real coefficients $h_{\mu p}$ and $h_{pp} \neq 0$ we shall have (§ 4.1)

$$(16.16) \qquad T = [A] = \prod_{p=0}^{l-1} h_{pp}^{-2} \cdot \prod_{p=0}^{l-1} \{(-1)^{p+n} W_p(u) W_{p+1}(u) x^{-n}\}.$$

---

6. We are concerned with the construction of certain orthogonal polynomials $\{h_p(z)\}$, $\{f_p(z)\}$, see below, without discussing the question of uniqueness.

4. We form first, $x = -u^{-1}$ is a parameter,

$$f_p(z) = \lambda_p(x^{-1}z) = \lambda_p(-uz). \tag{16.17}$$

These polynomials satisfy the "orthogonality relations"

$$K\{f_p(z)f_q(z)\} = \begin{cases} 0 & \text{if } p \neq q, \\ (-1)^p \binom{\alpha+2p-1}{p} \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+2p+1)} & \text{if } p = q. \end{cases} \tag{16.18}$$

Indeed, from (16.7), (16.13)

$$K\{f_p(z)z^\rho\} = \sum_{\nu=0}^{p} \binom{-\alpha-p}{p-\nu} \frac{1}{\nu!} \cdot K\{(-x^{-1}z)^{p-\nu} z^\rho\}$$

$$= \sum_{\nu=0}^{p} \binom{-\alpha-p}{p-\nu} \frac{(-1)^{p-\nu}}{\nu!} x^{\nu-p} \cdot (-x)^{p-\nu+\rho} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+p-\nu+\rho+1)}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+p+\rho+1)} (-x)^\rho \sum_{\nu=0}^{p} \binom{-\alpha-p}{p-\nu} \binom{\alpha+p+\rho}{\nu}$$

$$= \begin{cases} 0 & \text{if } \rho < p, \\ \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+p+\rho+1)} (-x)^\rho \dbinom{\rho}{p} & \text{if } \rho \geqq p, \end{cases} \tag{16.19}$$

The two latter equations follow from

$$(1+\xi)^{-\alpha-p}(1+\xi)^{\alpha+p+\rho} = (1+\xi)^\rho .$$

Now

$$f_p(z) = \binom{\alpha+2p-1}{p} (x^{-1}z)^p + \dots,$$

$$K\{[f_p(z)]^2\} = \binom{\alpha+2p-1}{p} x^{-p} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2p+1)} (-x)^p$$

$$= (-1)^p \binom{\alpha+2p-1}{p} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2p+1)}$$

so that (16.18) is valid.

5. Finally we proceed to the construction of the polynomials $h_p(z)$ (depending on the parameter x) which satisfy the relations (16.15). We write as in Christoffel's formula:

$$(1+z)^n h_p(z) =$$

$$(16.20) \quad = D_p \begin{vmatrix} f_p(z) & f_{p+1}(z) & \dots & f_{p+n}(z) \\ f_p(-1) & f_{p+1}(-1) & \dots & f_{p+n}(-1) \\ f'_p(-1) & f'_{p+1}(-1) & \dots & f'_{p+n}(-1) \\ \cdot & \cdot & \dots & \cdot \\ f_p^{(n-1)}(-1) & f_{p+1}^{(n-1)}(-1) & \dots & f_{p+n}^{(n-1)}(-1) \end{vmatrix}$$

$$= E_p \begin{vmatrix} f_p(z) & f_{p+1}(z) & \dots & f_{p+n}(z) \\ \lambda_p(u) & \lambda_{p+1}(u) & \dots & \lambda_{p+n}(u) \\ \lambda'_p(u) & \lambda'_{p+1}(u) & \dots & \lambda'_{p+n}(u) \\ \cdot & \cdot & \dots & \cdot \\ \lambda_p^{(n-1)}(u) & \lambda_{p+1}^{(n-1)}(u) & \dots & \lambda_{p+n}^{(n-1)}(u) \end{vmatrix}$$

where the real quantities D_p and E_p must be chosen properly, $D_p \neq 0$, $E_p \neq 0$. Of course D_p, E_p depend on p, n, α, x. Incidentally,

$$E_p = D_p \cdot x^{-1-2-\dots-n+1}.$$

In the second determinant appearing in (16.20) the factor of $f_p(z)$ is $W_{p+1}(u)$, and the factor of $f_{p+n}(z)$ is $(-1)^n W_p(u)$. We have $W_p(u) > 0$, $0 \leq p \leq l$. In view of (16.18) we conclude easily that (16.15) holds for $p \neq q$.

Now let us consider the case $p = q$. We have

$$h_p(z) = E_p \cdot (-1)^n W_p(u) \cdot x^{-p-n} \binom{\alpha+2p+2n-1}{p+n} z^p + \dots,$$

$$h_{pp} = E_p \cdot (-1)^n W_p(u) \cdot x^{-p-n} \binom{\alpha+2p+2n-1}{p+n},$$

$$K\{(1+z)^n [h_p(z)]^2\} = E_p \cdot (-1)^n W_p(u) \cdot x^{-p-n} \binom{\alpha+2p+2n-1}{p+n}$$

$$\cdot K\{(1+z)^n h_p(z) z^p\}$$

$$= E_p^2 \cdot (-1)^n W_p(u) W_{p+1}(u) \cdot x^{-p-n} \binom{\alpha+2p+2n-1}{p+n} \cdot K\{f_p(z) z^p\}$$

$$= E_p^2 \cdot (-1)^{p+n} W_p(u) W_{p+1}(u) \cdot x^{-p-n} \binom{\alpha+2p+2n-1}{p+n} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2p+1)} x^p.$$

Here we took (16.19) into account. Comparing this with (16.15) we find

$$1 = E_p^2 \binom{\alpha+2p+2n-1}{p+n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2p+1)} \tag{16.21}$$

so that E_p is indeed real and it depends on p, n, α (it is independent of x). Also

$$h_{pp}^{-2} = \left\{\binom{\alpha+2p+2n-1}{p+n}\right\}^{-1} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2p+1)} [W_p(u)]^{-2} x^{2p+2n}.$$

Consequently, using (16.16),

$$\begin{aligned} T = [A] &= \prod_{p=0}^{l-1} \left[\left\{\binom{\alpha+2p+2n-1}{p+n}\right\}^{-1} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2p+1)} \cdot (-1)^{p+n} \frac{W_{p+1}(u)}{W_p(u)} x^{2p+n}\right] \\ &= (-1)^{\frac{l(l-1)}{2}+ln} \cdot x^{l(l-1)+ln} \cdot \frac{W_l(u)}{W_0(u)} \cdot \prod_{p=0}^{l-1} \left[\left\{\binom{\alpha+2p+2n-1}{p+n}\right\}^{-1} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2p+1)}\right]. \end{aligned}$$

Since

$$W_0(u) = \lambda_0(u) \lambda'_1(u) \ldots \lambda_{n-1}^{(n-1)}(u) = \prod_{q=0}^{n-1} \left\{\binom{\alpha+2q-1}{q} q!\right\}$$

the Lemma will follow.

Of course, the restriction on x is unimportant.

6. The following remark is immaterial at the present moment but it will be useful later (§ 28.3). Let r be any integer, $r \geqq p$; we obtain from (16.19)

$$\begin{aligned} K\{(1+z)^r f_p(z)\} &= \sum_{j=0}^{r} \binom{r}{j} K\{f_p(z) z^j\} = \sum_{j=p}^{r} \binom{r}{j} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+p+j+1)} (-x)^j \binom{j}{p} \\ &= \binom{r}{p} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2p+1)} \cdot (-x)^p Q_{r-p}^{(\alpha+2p)}(x). \end{aligned} \tag{16.22}$$

§ 17. Theorem 5. Laguerre polynomials, l even: The Wronskian.

1. We prove now the following

Lemma 6: *Let l and n be positive integers, l even, $n \geqq 1$. We have then*

$$W(\lambda_l(u), \lambda_{l+1}(u), \ldots, \lambda_{l+n-1}(u)) = W_l(u) > 0 \tag{17.1}$$

provided u is negative, $\alpha > -1$; the polynomials $\lambda_p(u)$ are defined by (16.2), (16.7).

Under the assumption that l is even, we shall prove that the Wronskian $W_l(u)$ keeps a constant sign for $u < 0$; according to § 16.2 the leading term of $W_l(u)$ is (apart from trivial positive factors) u^{ln} so that the constant sign of $W_l(u)$ will be positive. Combining this with (16.1) and (16.4) the assertion of Theorem 5 will follow.

As explained at the beginning of § 16 the proof is based on the hypothesis that (17.1) holds for $\alpha = 0$. A proof of that fact was given in § 11.

We assume that $l \geqq 2$. Induction will be used with respect to n. For $n = 1$ we have, by using the second formula in (16.7):

$$\lambda_l(u) = \frac{1}{\Gamma(\alpha+l)\, l\,!} \sum_{\nu=0}^{l} \int_0^\infty e^{-t}\, t^{\alpha+l+\nu-1}\, dt \cdot \binom{l}{\nu} u^\nu$$

$$= \frac{1}{\Gamma(\alpha+l)\, l\,!} \int_0^\infty e^{-t}\, t^{\alpha+l-1} (1+tu)^l\, dt > 0$$

for all real u since l is even.

2. Now a similar reasoning follows as in the case of the ultraspherical polynomials so that we can be brief. As in § 15.2 we take out the coefficients of the highest powers $u^l, u^{l+1}, \ldots, u^{l+n-1}$ occuring in the first row of $W_l(u)$ and we find that

$$\prod_{m=l}^{l+n-1} \left\{ \binom{\alpha+2m-1}{m} \right\}^{-1} \cdot W_l(u) = \overline{W}(\alpha, u)$$

is a rational function of α, regular for $\alpha > -1$ (incl. $\alpha = \infty$), and a polynomial in u with a leading term which is the Wronskian of the powers mentioned. If $\varepsilon > 0$ is given, as in § 15 we find readily a positive number $v_0 = v_0(l, n, \varepsilon)$ such that $\overline{W} > 0$ holds for $u < 0$, $|u| \geqq v_0$ and for all α, $\alpha \geqq -1 + \varepsilon$.

3. The further part of the argument centers around the quantity $\min \overline{W} = m(\alpha)$ in the finite interval $u \leqq 0$, $|u| \leqq v_0$; a slight modification

will be necessary due to the point $u=0$. We show first that $W_l(u)$ can not vanish for $u=0$ where α is arbitrary.

It is convenient to conclude this by making use of the identity (16.1) proved above and discussing the determinant T occuring on the left of (16.1) for large values of x. Indeed, $\lim_{x\to\infty} x^{-l(l-1)-ln}T$ agrees with $W_l(0)$ apart from a constant factor different from zero. In view of (16.10) the leading term of $L_n^{(\alpha)}(x)/L_n^{(\alpha)}(0) = Q_n^{(\alpha)}(x)$ is

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}(-x)^n$$

so that the leading term of T will be given by the determinant

$$\left[\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+\mu+\nu+1)}(-x)^{n+\mu+\nu}\right]_0^{l-1}, \quad \mu, \nu = 0, 1, \ldots, l-1;$$

hence $(-x)^{-l(l-1)-ln}T$ tends to

$$\left[\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+\mu+\nu+1)}\right]_0^{l-1}$$

as $x\to\infty$. Taking out the elements of the last column we obtain after appropriate combination of the columns, the Vandermondian of the numbers $0, 1, \ldots, l-1$; the columns appear in the reverse order. Thus the limit is $\neq 0$ and $W_l(0)\neq 0$.

Starting out from the fact that $m(0)>0$ (this was proved in § 11) we deal with the hypothesis $\overline{W}=0$, $\dfrac{\partial\overline{W}}{\partial u}=0$ for a certain α and u; we can assume now that $u<0$. In the further course we use the operator [cf. (16.8)]

$$L\lambda_p = u^2\lambda_p'' + [1+(\alpha+1)u]\lambda_p'$$

so that $L\lambda_p = p(\alpha+p)\lambda_p$. We observe also that in view of (16.8), λ_p and λ'_p can not vanish at the same time, $u\neq 0$.

The rest of the argument is quite similar to that in § 15.

We proved in fact more, namely:

The determinant $W_l(u)$ in (17.1) is positive for all real u if l is even; it is positive for $u\geqq 0$ if l is odd.

Thus, the quantity $(-1)^{l/2}T$ is positive for all real x if l is even

[except for $x=0$ where it has a zero of order $l(l-1)$]; moreover, $(-1)^{(l-1)/2}T$ is positive for $x<0$ if l is odd. (Cf. (32.8).)

§ 18. Theorem 5, Hermite polynomials.

1. This case can be settled in a simple way. We prove

Lemma 7: If $H_n(x)$ is the n^{th} Hermite polynomial, we have the following matrix identity:

$$(18.1)\qquad (H_{n+\nu}^{(\mu)}(x))=(\lambda_{\mu\nu}(x))\,(H_{n+\mu+\nu}(x)),\quad \mu,\nu=0,1,\ldots,l-1,$$

where $\lambda_{\mu\nu}(x)$ are certain polynomials depending on μ,ν but not on n; moreover $\lambda_{\mu\nu}(x)=0$ for $\mu<\nu$, $\lambda_{\mu\mu}(x)=(-1)^{\mu}$. Passing to the determinants we obtain the relation

$$(18.2)\qquad W(H_n(x), H_{n+1}(x),\ldots, H_{n+l-1}(x))=(-1)^{\frac{l(l-1)}{2}}\cdot$$
$$\cdot T(H_n(x), H_{n+1}(x),\ldots, H_{n+l-1}(x)).$$

Let l be even. Applying Theorem 1 to $H_n(-x)$ we see that the Wronskian on the left is positive for all real values of x. Hence $\operatorname{sgn} T=(-1)^{l/2}$ follows from (18.2) immediately.

2. The proof of (18.1) is based on the simple identity (5.15), second part. Repeated application of this formula leads to

$$(18.3)\qquad H_n^{(\mu)}(x)=\sum_{k=0}^{\mu}\lambda_{\mu k}(x)\,H_{n+k}(x)$$

where the polynomials $\lambda_{\mu\nu}(x)$ have all the properties mentioned in the Lemma; $\lambda_{\mu\mu}(x)=(-1)^{\mu}$ follows by induction. Hence

$$H_{n+\nu}^{(\mu)}(x)=\sum_{k=0}^{l-1}\lambda_{\mu k}(x)\,H_{n+k+\nu}(x)$$

which is identical with (18.1).

This completes the proof of Theorem 5.

§ 19. Theorem 6, classical polynomials, l odd.

We present the proof of Theorem 6 for Legendre polynomials. The proof for the other cases does not offer any new difficulties. We follow

the instructions of § 2 taking the identity (5.2) into account. Essential use will be made of Theorem 5.

1. We employ the abbreviation

$$T\,(P_n(x)\,,\,P_{n+1}(x),\,\ldots,\,P_{n+l-1}(x)) = f_n(x)$$

where l is a fixed odd number. We verify easily that

$$f_n(-x) = (-1)^{ln} f_n(x) = (-1)^n f_n(x)\,. \tag{19.1}$$

We know [cf. (4.9)] that $f_0(x) \neq 0$ in $-1 < x < +1$.

As a further preparation we establish the sign of $f_n(x)$ in the left neighborhood of $x = 1$. Conveniently we use Lemma 1 of § 12; if x is near 1, u large, the leading term of W is (apart from trivial positive factors) $u^{ln} = \left(-\dfrac{x}{\sqrt{x^2-1}}\right)^{ln}$ so that for $x = 1 - \varepsilon$, $\varepsilon > 0$,

$$\operatorname{sgn} f_n(x) = \operatorname{sgn}\left[(-1)^{ln} \cdot (x^2-1)^{\frac{l(l-1)}{2}} \cdot (-x)^{ln}\right] = (-1)^{\frac{l(l-1)}{2}} = (-1)^{\frac{l-1}{2}}.$$

Thus all $f_n(x)$ have the same sign for $x = 1 - \varepsilon$ and the signs of $f_n(x)$ alternate for $x = -1 + \varepsilon$ [by (19.1)].

2. It remains to prove the following:

No two consecutive functions $f_n(x)$ can vanish for the same x in $-1 < x < +1$. More precisely, if $f_n(x) = 0$, we have

$$f_{n-1}(x)\, f_{n+1}(x) < 0. \tag{19.2}$$

Also for the same x

$$f_{n-1}(x)\, f_n'(x) > 0 \tag{19.3}$$

so that the zeros of $f_n(x)$ are all simple.

Inequality (19.2) follows easily by applying Sylvester's theorem to the first and last rows and columns of the determinant $T\,(P_{n-1}, P_n, \ldots, P_{n+l-1})$ where $P_n = P_n(x)$. We obtain

$$T\,(P_{n-1}\,,\,P_n\,,\,\ldots\,,\,P_{n+l-1}) \cdot T\,(P_{n+1}\,,\,P_{n+2}\,,\,\ldots\,,\,P_{n+l-1})$$
$$= \begin{vmatrix} f_{n-1}(x) & f_n(x) \\ f_n(x) & f_{n+1}(x) \end{vmatrix}.$$

In view of Theorem 5 the left hand product is of the sign

$$(-1)^{(l+1)/2} \cdot (-1)^{(l-1)/2} = (-1)^l < 0$$

so that

$$f_{n-1}(x) f_{n+1}(x) - [f_n(x)]^2 < 0, \quad -1 < x < +1. \tag{19.4}$$

This yields (19.2)

3. We pass now to the essential part of the argument which is the proof of (19.3). For this purpose we note first that certain real numbers $a_0, a_1, \ldots, a_{l-1}$ exist not all zero such that

$$a_0 P_{n+\mu} + a_1 P_{n+\mu+1} + \ldots + a_{l-1} P_{n+\mu+l-1} = 0, \quad \mu = 0, 1, \ldots, l-1. \tag{19.5}$$

The two extreme constants a_0, a_{l-1} can not be zero, otherwise a determinant of type T and of even order $l-1$ would vanish which contradicts Theorem 5. Let $a_0 = 1$. We form $f'_n(x)$ by differentiating the individual columns of $f_n(x)$ and adding the resulting determinants. Indicating these determinants by the elements of their first row we have

$$f_n'(x) = \sum_{\nu=0}^{l-1} (P_n, P_{n+1}, \ldots, P_{n+\nu-1}, P'_{n+\nu}, P_{n+\nu+1}, \ldots, P_{n+l-1}).$$

We multiply the columns of number $1, 2, \ldots, l-1$, except the column of number ν, by $a_1, a_2, \ldots, a_{l-1}$ respectively; adding them to the column of number 0 we obtain in the first row, in view of (19.5),

$$\begin{aligned} &(-a_\nu P_{n+\nu}, P_{n+1}, \ldots, P_{n+\nu-1}, P'_{n+\nu}, P_{n+\nu+1}, \ldots, P_{n+l-1}) \\ &= (a_\nu P'_{n+\nu}, P_{n+1}, \ldots, P_{n+\nu-1}, P_{n+\nu}, P_{n+\nu+1}, \ldots, P_{n+l-1}) \end{aligned}$$

so that

$$f'_n(x) = \sum_{\nu=0}^{l-1} (a_\nu P'_{n+\nu}, P_{n+1}, \ldots, P_{n+l-1}).$$

Multiplying further the single rows of number $1, 2, \ldots, l-1$ by $a_1, a_2, \ldots, a_{l-1}$ and adding them to the first row (of number 0) we obtain in the first row

$$\left(\sum_{\mu,\nu=0}^{l-1} a_\mu a_\nu P'_{n+\mu+\nu}, 0, \ldots, 0\right);$$

here the minor of the first element is $T(P_{n+2}, \dots, P_{n+l})$ which is of even order $l-1$, hence of the sign $(-1)^{(l-1)/2}$. Thus

$$\operatorname{sgn} f_n'(x) = (-1)^{\frac{l-1}{2}} \cdot \operatorname{sgn} \sum_{\mu,\nu=0}^{l-1} a_\mu a_\nu P'_{n+\mu+\nu}. \tag{19.6}$$

4. We discuss now

$$f_{n-1}(x) = T(P_{n-1}, P_n, \dots, P_{n+l-2}).$$

Multiplying again the columns of number 1, 2, ..., $l-1$ by $a_1, a_2, \dots, a_{l-1}$ and adding them to the first column we obtain in the first row:

$$\left(\sum_{\nu=0}^{l-1} a_\nu P_{n+\nu-1}, P_n, \dots P_{n+l-2}\right);$$

the further elements in the first column are all 0. The minor of the first element of the first row is $T(P_{n+1}, \dots, P_{n+l-1})$ the sign of which is $(-)^{(l-1)/2}$ so that (19.3) is equivalent to the following inequality:

$$\sum_{\mu,\nu=0}^{l-1} a_\mu a_\nu (1-x^2) P'_{n+\mu+\nu} \cdot \sum_{\nu=0}^{l-1} a_\nu P_{n+\nu-1} > 0. \tag{19.7}$$

We note that the second sum is different from zero, otherwise $f_{n-1}(x)$ would vanish in contradiction to (19.2).

5. With the aid of the identity (5.2) we write the first factor in (19.7) as follows:

$$\sum_{\mu,\nu=0}^{l-1} a_\mu a_\nu (n+\mu+\nu)(P_{n+\mu+\nu-1} - xP_{n+\mu+\nu}) =$$

$$= \sum_{\nu=0}^{l-1} (n+\nu) a^\nu \left\{\sum_{\mu=0}^{l-1} a_\mu P_{n+\mu+\nu-1} - x \sum_{\mu=0}^{l-1} a_\mu P_{n+\mu+\nu}\right\}$$

$$+ \sum_{\mu=1}^{l-1} \mu a_\mu \left\{\sum_{\nu=0}^{l-1} a_\nu P_{n+\mu+\nu-1} - x \sum_{\nu=0}^{l-1} a_\nu P_{n+\mu+\nu}\right\}.$$

The sums in the curly brackets are 0 in view of (19.5) except the first for $\nu = 0$ so that we obtain for (19.7):

$$na_0 \sum_{\mu=0}^{l-1} a_\mu P_{n+\mu-1} \cdot \sum_{\nu=0}^{l-1} a_\nu P_{n+\nu-1} = n \left(\sum_{\mu=0}^{l-1} a_\mu P_{n+\mu-1} \right)^2 > 0.$$

Thus the assertion follows.

6. No new difficulties arise in the other cases of Theorem 6. We must use the following formulas: (5.6); (5.14); (5.15). Moreover we employ the identities (14.1), (14.2), (14.3), and the remarks in § 14.4; also (16.1), (16.4), (16.5) and § 17.3.

§ 20. Theorem 7, special discrete measures, $l = 2$.

In this section we follow the notation introduced in § 5.3. The proof of Theorem 7 is very simple in the cases (a), (b), (c) and somewhat more involved in the case (d). In all cases we form a suitable generating function and prove that it belongs to the Laguerre-Pólya-I.Schur class (or in the finite case it is a polynomial with only real zeros), provided x is on the spectrum.

(a) Poisson-Charlier polynomials:

$$Q_n(x) = c_n(a\,;x) = c_n(x).$$

The following generating function holds:

$$\sum_{n=0}^{\infty} \frac{c_n(x)}{n!} z^n = e^z \left(1 - \frac{z}{a}\right)^x \qquad [13, (2.81.3)];^{(7)} \tag{20.1}$$

this formula is valid for any x, integer or not. If x is a positive integer, the function on the right will be indeed of the Laguerre-Pólya-I.Schur class. This yields the assertion of Theorem 7.

(b) Meixner polynomials:

$$Q_n(x) = M_n(\beta, \gamma\,; x) = M_n(x).$$

The following generating function holds:

7. Our $c_n(a\,;x)$ is identical with Erdélyi's $c_n(x\,;a)$ [2, Vol. 2, p. 226]. Formula (6), loc. cit. must be corrected, replacing e^{-z} on the right by e^z.

$$\sum_{n=0}^{\infty} \frac{M_n(x)}{n!} z^n = e^z \, {}_1F_1\left(-x ; \beta ; \left(\frac{1}{\gamma} - 1\right) z\right) \tag{20.2}$$

where ${}_1F_1$ is the standard notation for the confluent hypergeometric series [13, (5.3.1), (5.3.3)]. If x is a positive integer, the right hand expression becomes

$$\left\{\binom{x+\beta-1}{x}\right\}^{-1} \cdot e^z L_x^{(\beta-1)}(z(1/\gamma - 1)) \tag{20.3}$$

and this function of z is again of the Laguerre-Pólya-I.Schur class.

For the proof of (20.2) we note that the left hand side of (20.2) can be written as a double sum as follows:

$$\sum \frac{(-n)_\nu (-x)_\nu}{\nu!\,(\beta)_\nu} \left(1 - \frac{1}{\gamma}\right)^\nu \cdot \frac{z^n}{n!}, \quad n, \nu = 0,1,2, \ldots ; \quad n \geqq \nu,$$

where we follow the notation:

$$(a)_0 = 1 \; ; \; (a)_\nu = a(a+1)(a+2)\ldots(a+\nu-1).$$

Now we carry out first the summation with respect to n, $n \geqq \nu$, and then with respect to ν. This yields the formula (20.2) without difficulty.

(c) Krawtchouk polynomials:

$$Q_n(x) = k_n(x) / k_n(0).$$

The following (finite) generating function holds:

$$\sum_{n=0}^{N} \binom{N}{n} \frac{k_n(x)}{k_n(0)} z^n = (1+z)^{N-x} \left(1 - \frac{q}{p} z\right)^x, \qquad x = 0,1,2, \ldots, N. \tag{20.4}$$

Indeed,

$$\begin{aligned}\sum_{n=0}^{N} k_n(x) \left(\frac{z}{p}\right)^n &= \sum_{\nu, n} (-1)^\nu \binom{N-x}{n-\nu} \binom{x}{\nu} \left(\frac{q}{p}\right)^\nu z^n \\ &= \sum_{\nu=0}^{x} \left\{ \sum_{n=\nu}^{N-x+\nu} \binom{N-x}{n-\nu} z^{n-\nu} \right\} (-1)^\nu \binom{x}{\nu} \left(\frac{q}{p}\right)^\nu z^\nu\end{aligned}$$

and this yields (20.4) as well as the assertion of Theorem 7.

Replacing $\binom{N}{n} \dfrac{1}{k_n(0)}$ by p^{-n} in (20.4) and extending the summation

from $n=0$ to $n=\infty$, the same generating function arises as in (20.4); here x is arbitrary.

(d) Tchebychev's polynomials of a discrete measure:

$$Q_n(x) = t_n(x) / t_n(0).$$

The polynomials $\{t_n(x)\}$ are defined by (5.29), cf. also (5.31). We prove the following generating function:

$$(20.5)\quad G(x,z) = \sum_{n=0}^{N-1} \binom{N-1}{n} \frac{t_n(x)}{t_n(0)} (-z)^n = \sum_{n=0}^{N-1} \frac{t_n(x)}{n!} z^n$$
$$= (1-z)^{p-x} \frac{f^{(p)}(1-z^2)}{p!}$$

where x is an integer, $x = 0, 1, \ldots, N-1$, and

$$(20.6)\qquad f(u) = u^x (u-1)^p, \quad p = N-1-x.$$

The right hand side can also be expressed in terms of certain Jacobi polynomials; see below.

For the proof of (20.5) let us consider an arbitrary positive (not necessarily integral) value of x, a fixed positive integer N, and let $0<\rho<1$. In view of (5.29) we form the following infinite series:

$$\sum_{n=0}^{\infty} \frac{t_n(x)}{n!} z^n \rho^n = \sum\sum (-1)^{n-\nu} \binom{n}{\nu} \binom{x+\nu}{n} \binom{x+\nu-N}{n} z^n \rho^n$$
$$= \sum\sum (-1)^{n-\nu} \binom{x+\nu}{\nu} \binom{x}{n-\nu} \binom{x+\nu-N}{n} z^n \rho^n$$

where the range of the summation is $n \geqq 0$, $\nu \geqq 0$, $n \geqq \nu$. This series is absolutely convergent provided $|z|$ is sufficiently small $\left(\text{say } |z| < \frac{1}{32}\right)$ since

$$\binom{n}{\nu} \leqq 2^n, \quad \left|\binom{x+\nu}{n}\right| \leqq \binom{x+\nu+n-1}{n} \leqq \binom{2n+a}{n} \leqq 2^{2n+a},$$
$$\left|\binom{x+\nu-N}{n}\right| \leqq \binom{2n+N+a}{n} \leqq 2^{2n+N+a}, \qquad a = [x].$$

Hence we have for $|z| < \frac{1}{32}$ and for $0<\rho<1$,

$$(20.7)\quad \sum_{n=0}^{\infty}\frac{t_n(x)}{n!}z^n\rho^n=\sum_{\nu=0}^{\infty}\binom{x+\nu}{\nu}z^\nu\cdot\sum_{n=\nu}^{\infty}\binom{x}{n-\nu}(-z)^{n-\nu}\binom{x+\nu-N}{n}\rho^n$$

$$=\sum_{\nu=0}^{\infty}\binom{x+\nu}{\nu}z^\nu\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-ze^{-i\vartheta})^x(1+\rho e^{i\vartheta})^{x+\nu-N}e^{-i\nu\vartheta}\,d\vartheta\,.$$

Indeed for a fixed ν, the sum $\sum\limits_{n=\nu}^{\infty}$ can be written as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=\nu}^{\infty}\binom{x}{m-\nu}(-ze^{-i\vartheta})^{m-\nu}\cdot\sum_{n=\nu}^{\infty}\binom{x+\nu-N}{n}(\rho e^{i\vartheta})^n\cdot e^{-i\nu\vartheta}\,d\vartheta$$

and to the latter sum we may add all terms $n<\nu$. Performing the summation in (20.7) with respect to ν,

$$\sum_{n=0}^{\infty}\frac{t_n(x)}{n!}z^n\rho^n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{1-ze^{-i\vartheta}(1+\rho e^{i\vartheta})\}^{-x-1}(1-ze^{-i\vartheta})^x(1+\rho e^{i\vartheta})^{x-N}\,d\vartheta\,.$$

The summation and integration are interchangeable since

$$|z(1+\rho e^{i\vartheta})|<2|z|<\frac{1}{16}\,.$$

The right hand expression can be written as follows:

$$(20.8)\qquad \frac{1}{2\pi i}\int(1-z\rho-z\zeta^{-1})^{-x-1}(1-z\zeta^{-1})^x(1+\rho\zeta)^{x-N}\frac{d\zeta}{\zeta}\,.$$

Here $|z|<\frac{1}{32}$, $0<\rho<1$, and the integration is extended over the unit circle $|\zeta|=1$ in the positive direction. The meaning of the multi-valued functions is clear.

Now let x be an integer, $0\leqq x\leqq N-1$. Then $t_n(x)=0$ for $n\geqq N$ (cf. the remark to (5.31)) and the left hand sum terminates with $n=N-1$. The integrand of (20.8) will be single-valued. We integrate over $|\zeta|=1$ in the positive direction, with an indentation at $\zeta=-1$; afterwards we may allow ρ to approach 1 and obtain the following formula:

$$\sum_{n=0}^{N-1}\frac{t_n(x)}{n!}z^n=\frac{1}{2\pi i}\int[(1-z)\zeta-z]^{-x-1}(\zeta-z)^x(1+\zeta)^{x-N}\,d\zeta\,.$$

Since $|z| < \frac{1}{32}$ we have $\left|\frac{z}{1-z}\right| < \frac{1}{31}$. For the contour of integration any closed curve might be chosen which includes $\frac{z}{1-z}$ and excludes -1. Now the integrand vanishes at $\zeta = \infty$ like ζ^{-1+x-N}, $-1+x-N \leqq -2$ so that we can replace the contour by a small circle about $\zeta = -1$ described in the negative direction. Let $z \neq 0$. Introducing the new variable u by

$$\zeta - z = -\frac{z^2 u}{(1-z)(1-u)}, \quad (1-z)\zeta - z = -\frac{z^2}{1-u},$$

$$1+\zeta = \frac{1-z^2-u}{(1-z)(1-u)}, \quad d\zeta = -\frac{z^2}{1-z}\,\frac{du}{(1-u)^2},$$

the point $\zeta = -1$ is transformed into $u = 1-z^2$. The direction of the contour is reversed and we obtain

$$(20.9) \qquad \sum_{n=0}^{N-1} \frac{t_n(x)}{n!} z^n = (1-z)^{N-1-2x} \cdot \frac{1}{2\pi i} \int \frac{u^x(1-u)^{N-1-x}}{(1-z^2-u)^{N-x}}\, du$$

where the contour encircles $1-z^2$ in the negative direction.

This yields the generating function (20.5).

Clearly, the function $f(u)$ defined by (20.6) is a polynomial of degree $N-1$ in u so that the right hand expression in (20.5) is indeed a polynomial in z of the exact degree $p-x+2(N-1-p) = N-1$.

The right hand side can be expressed in terms of certain Jacobi polynomials by using Rodrigues' formula [13, (4.3.1) and (4.22.2)]. An easy calculation leads to the following interesting representations:

$$(20.10) \quad \sum_{n=0}^{N-1} \frac{t_n(x)}{n!} z^n = \begin{cases} (-1)^{N-1-x}(1+z)^{2x-N+1} \cdot P_{N-1-x}^{(2x-N+1,\,0)}(2z^2-1) \\ (-1)^x (1-z)^{N-1-2x} \cdot P_x^{(N-1-2x,\,0)}(2z^2-1). \end{cases}$$

Here $x = 0, 1, 2, \ldots, N-1$ and both representations can be used.

In view of familiar properties of the zeros of the Jacobi polynomials we conclude that for a fixed integer x, $0 < x < N-1$, the generating function (20.10) as a polynomial in z has only real zeros. Indeed, in the first case $x \geqq \frac{N-1}{2}$ (use the first formula) we have $z = -1$ as a zero of order

$2x-N+1$ and in addition $N-1-x$ pairs of zeros different from zero and symmetrically located in $-1<z<1$; similarly for $x \leqq \frac{N-1}{2}$ (use the second formula) we have $z=1$ as a zero of order $N-1-2x$ and x pairs of zeros not vanishing and symmetrically located in $-1<z<1$. Thus all zeros are real and they are not all the same. This establishes the inequality of Theorem 7.

§ 21. A discrete analog of the Jacobi polynomials.

Concerning the finite system of orthogonal polynomials discussed in this section we refer to W. Hahn [6], Erdélyi [cf.2, Vol.2, pp.223-224], and [15]. The polynomials $t_n(a, b; x)$ where $a>-1$, $b>-1$, defined below generalize Tchebychev's discrete measure polynomials $t_n(x)$, cf. (5.29); they arise for $a=b=0$.

From the polynomials $t_n(a, b; x)$, by an appropriate limiting process, Jacobi polynomials arise [cf. (21.8)]; for this reason Turán's inequality ($l=2$) can not hold for $t_n(a, b; x)$ unrestrictedly, see § 3.3. However we shall prove that it holds if $a=b$; this corresponds of course to the ultraspherical case.

1. Definitions. We consider following Erdélyi:

(21.1) $$\beta = 1+a\,, \quad \gamma = 1-N\,, \quad \delta = 1-N-b$$

where $a>-1$, $b>-1$, N a positive integer. The jump function will be [loc. cit. (11)]

(21.2) $$j(a, b; x) = \frac{(\beta)_x(\gamma)_x}{x!\,(\delta)_x}$$

which is positive on the spectrum $x=0,1,\dots,N-1$. The following representations hold:

$$(21.3)\qquad j(a\,,b\,;x)=\begin{cases}\dfrac{\dbinom{x+a}{a}\dbinom{N-1-x+b}{b}}{\dbinom{N-1+b}{b}}=\dfrac{\dbinom{x+a}{a}\dbinom{x-N}{b}}{\dbinom{-N}{b}},\\[2ex] \dfrac{\dbinom{x+a}{x}\dbinom{N-1-x+b}{N-1-x}}{\dbinom{N-1+b}{N-1}},\\[2ex] \dfrac{(N-1)!}{\Gamma(a+1)\,\Gamma(N+b)}\cdot\dfrac{\Gamma(x+a+1)\,\Gamma(N-x+b)}{\Gamma(x+1)\,\Gamma(N-x)}.\end{cases}$$

The two representations appearing in the first row are convenient if a and b are integers, x arbitrary; the second row is convenient if x is an integer, $0\leqq x\leqq N-1$, but a and b are arbitrary; the third row holds unrestrictedly for $a\,,b\,,x$.

Further we form, following Erdélyi [loc. cit. (8)]:

$$(21.4)\qquad \frac{(\beta)_x\,(\gamma)_x}{(x-n)!\,(\delta)_{x-n}}=\begin{cases}(-1)^n\dfrac{\dbinom{n+a}{a}\dbinom{n+b}{b}n!^2}{\dbinom{N-1+b}{b}}\dbinom{x+a}{n+a}\dbinom{N-1-x+n+b}{n+b},\\[2ex] (-1)^n\dfrac{\dbinom{n+a}{n}\dbinom{n+b}{n}n!^2}{\dbinom{N-1+b}{N-1}}\dbinom{x+a}{x-n}\dbinom{N-1-x+n+b}{N-1-x},\\[2ex] \dfrac{(-1)^n(N-1)!}{\Gamma(a+1)\,\Gamma(N+b)}\cdot\dfrac{\Gamma(x+a+1)\,\Gamma(N-x+n+b)}{\Gamma(x-n+1)\,\Gamma(N-x)},\end{cases}$$

again corresponding to the three cases mentioned above. Also the last factor of the first formula might be replaced by $(-1)^{n+b}\dbinom{x-N}{n+b}$. Hence we can define the orthogonal polynomials $t_n(a\,,b\,;x)$ associated with (21.3) as follows (the proof of the orthogonality is indicated below):

$$(21.5)\quad \begin{cases} \binom{x+a}{a}\binom{N-1-x+b}{b} t_n(a\,,b\,;x) \\ \qquad = (-1)^n \binom{n+a}{a}\binom{n+b}{b} n\,!\cdot\Delta^n \binom{x+a}{n+a}\binom{N-1-x+n+b}{n+b}, \\ \binom{x+a}{x}\binom{N-1-x+b}{N-1-x} t_n(a\,,b\,;x) \\ \qquad = (-1)^n \binom{n+a}{n}\binom{n+b}{n} n\,!\cdot\Delta^n \binom{x+a}{x-n}\binom{N-1-x+n+b}{N-1-x}, \\ \dfrac{\Gamma(x+a+1)\,\Gamma(N-x+b)}{\Gamma(x+1)\,\Gamma(N-x)} t_n(a\,,b\,;x) \\ \qquad = \dfrac{(-1)^n}{n\,!}\Delta^n \dfrac{\Gamma(x+a+1)\,\Gamma(N-x+n+b)}{\Gamma(x-n+1)\,\Gamma(N-x)} \\ \qquad = \dfrac{(-1)^n}{n\,!}\displaystyle\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}\dfrac{\Gamma(x+\nu+a+1)\,\Gamma(N-x-\nu+n+b)}{\Gamma(x+\nu-n+1)\,\Gamma(N-x-\nu)}\,. \end{cases}$$

The polynomials $t_n(a\,,b\,;x)$ are identical with $p_n(x\,;\beta\,,\gamma\,,\delta)$ [loc. cit. (8)] where (21.1) holds.

Now

$$\left\{\frac{\Gamma(x+a+1)\,\Gamma(N-x+b)}{\Gamma(x+1)\,\Gamma(N-x)}\right\}^{-1}\cdot\frac{\Gamma(x+\nu+a+1)\,\Gamma(N-x-\nu+n+b)}{\Gamma(x+\nu-n+1)\,\Gamma(N-x-\nu)}$$
$$=\frac{\Gamma(x+\nu+a+1)}{\Gamma(x+a+1)}\cdot\frac{\Gamma(N-x-\nu+n+b)}{\Gamma(N-x+b)}\cdot\frac{\Gamma(x+1)}{\Gamma(x+\nu-n+1)}\cdot\frac{\Gamma(N-x)}{\Gamma(N-x-\nu)};$$

the first factor is a polynomial of degree ν in $x+a$, the second one is of degree $n-\nu$ in $-x+b$, the third one is of degree $n-\nu$ in x, the fourth one is of degree ν in x. Hence $t_n(a\,,b\,;x)$ is a polynomial in $a\,,b\,,x$ (in particular of degree n in x in view of the first expression (21.5)). Also $t_n(a\,,a\,;x)$ is of degree n in a.

From the second formula (21.5) we conclude

$$(21.6)\qquad t_n(a\,,b\,;0)=(-1)^n n\,!\binom{N-1}{n}\cdot\binom{n+a}{n},\qquad n=0,1,\ldots,N-1.$$

For the sake of completeness we prove briefly the orthogonality relation

$$(21.7)\qquad \sum_{x=0}^{N-1} j(a\,,b\,;x)\,t_n(a\,,b\,;x)\cdot x^\rho=0\,,\qquad \rho=0,1,\ldots,n-1\,.$$

Conveniently we start with the second formula in (21.5). Since

$$\binom{n+a}{n}\binom{n+b}{n}\binom{x+\nu+a}{x+\nu-n}\binom{N-1-x-\nu+n+b}{N-1-x-\nu}, \quad \begin{array}{l} x=0,1,\ldots,N-1, \\ \nu=0,1,\ldots,n, \end{array}$$

is a polynomial in a and b it suffices to prove (21.7) for non-negative integers a and b. So we can use the first formula in (21.5) and proceed as usual. We employ the identity $[f(x), g(x)$ polynomials$]$

$$\sum_{x=0}^{N-1} \Delta f(x) \, . \, g(x) = -\sum_{x=0}^{N-1} f(x) \, . \, \Delta g(x-1) + f(N)\, g(N-1) - f(0)\, g(-1)$$

n times. Since

$$\binom{x+\nu+a}{n+a}\binom{N-1-x-\nu+n+b}{n+b}, \qquad 0 \leqq \nu \leqq n-1,$$

vanishes for $x=0$ and $x=N$ the conclusion follows easily.

Finally, for fixed a, b, x, n we have

$$\lim_{N\to\infty} N^{-n} t_n(a, b; Nx) = (-1)^n P_n^{(a,b)}(1-2x) \tag{21.8}$$

where $P_n^{(a,b)}(x)$ is Jacobi's polynomial. The proof of this formula is based on the same argument as in [13, p. 34] for the case $a=b=0$. It is convenient to use the formula in the third row of (21.5). [8]

2. Generating function. The generating function (20.5) (which is the "Legendre case") can be extended to the "ultraspherical case". We prove the identity

$$\begin{aligned} &\binom{N-1-x+a}{N-1-x}\sum_{n=0}^{N-1}\binom{N-1}{n}\frac{t_n(a, a; x)}{t_n(a, a; 0)}(-z)^n \\ &\qquad = (-z^2)^{-a}(1-z)^{p-x}\frac{f^{(p)}(1-z^2)}{p!}, \\ &f(u) = u^x(u-1)^{p+a}, \quad p = N-1-x, \end{aligned} \tag{21.9}$$

where x is an integer, $0 \leqq x \leqq N-1$, $a > -1$. Let $0 < x < N-1$; the polynomial in z appearing on the right has only real zeros not all of which

8. Cf. [2, Vol. 2, p. 224, formula (13)] which must be corrected by deleting the factor $\binom{n+\alpha}{\alpha}$ on the right.

coincide (see below). From this fact Turán's inequality (in the 2×2 case) follows immediately.

Since $t_n(a, a; x)$ is a polynomial in a and in view of (21.6), the expression on the left of (21.9) is a rational function of a. The right hand side is visibly a polynomial in a. Thus it suffices to prove (21.9) for integer a and in this case we can use the first formula in (21.5). The argument of § 20 (d) requires only a slight modification so that we will be brief.

First let x be arbitrary positive. For sufficiently small $|z|$ and $0<\rho<1$ we have

$$\binom{x+a}{a}\binom{N-1-x+a}{a}\sum_{n=0}^{\infty}\frac{t_n(a, a; x)}{n!\binom{n+a}{a}}z^n\rho^n$$

$$=(-1)^a\sum_{n\geqq\nu}\binom{n+a}{a}\cdot(-1)^{n-\nu}\binom{n}{\nu}\binom{x+\nu+a}{n+a}\binom{x+\nu-N}{n+a}z^n\rho^n$$

$$=(-1)^a\binom{x+a}{a}\sum_{n\geqq\nu}(-1)^{n-\nu}\binom{x+\nu+a}{\nu}\binom{x}{n-\nu}\binom{x+\nu-N}{n+a}z^n\rho^n$$

$$=(-1)^a\binom{x+a}{a}\sum_{\nu=0}^{\infty}\binom{x+\nu+a}{\nu}z^\nu\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=\nu}^{\infty}\binom{x}{m-\nu}(-ze^{-i\vartheta})^{m-\nu}$$

$$\cdot\sum_{n=\nu}^{\infty}\binom{x+\nu-N}{n+a}(\rho e^{i\vartheta})^{n+a}\cdot\rho^{-a}e^{-i(\nu+a)\vartheta}\,d\vartheta,$$

and to the last sum we can add all terms with $-a\leqq n<\nu$. Hence

$$\binom{N-1-x+a}{a}\sum_{n=0}^{\infty}\frac{t_n(a, a; x)}{n!\binom{n+a}{a}}z^n\rho^n \tag{21.10}$$

$$=\frac{(-1)^a}{2\pi}\int_{-\pi}^{\pi}\{1-ze^{-i\vartheta}(1+\rho e^{i\vartheta})\}^{-x-a-1}(1-ze^{-i\vartheta})^x(1+\rho e^{i\vartheta})^{x-N}\rho^{-a}e^{-ia\vartheta}\,d\vartheta.$$

Now let x be an integer, $0\leqq x\leqq N-1$. From the first formula in (21.5) we conclude that $t_n(a, a; x)=0$ for $n\geqq N$, and the left hand sides of (21.9) and (21.10) will be identical as $\rho\to 1$. Hence we obtain as in § 20 (d):

$$\frac{(-1)^a}{2\pi i}\int \{(1-z)\zeta - z\}^{-x-a-1}(\zeta-z)^x(1+\zeta)^{x-N}\,d\zeta$$
$$= z^{-2a}(1-z)^{N-1-2x}\cdot\frac{1}{2\pi i}\int \frac{u^x(1-u)^{N-1-x+a}}{(1-z^2-u)^{N-x}}\,du$$

where ζ and u have the same meaning as there. This establishes the identity (21.9).

As in (20.10) the following alternative expressions for (21.9) can be obtained:

$$(21.11)\qquad \begin{aligned}&(-1)^{N-1-x}(1+z)^{2x-N+1}P_{N-1-x}^{(2x-N+1,\,a)}(2z^2-1)\\ &\qquad = (-1)^x\frac{\binom{N-1+a}{N-1-x}}{\binom{N-1+a}{x}}(1-z)^{N-1-2x}P_x^{(N-1-2x,\,a)}(2z^2-1)\end{aligned}$$

where $P_m^{(\alpha,\beta)}(t)$ is Jacobi's polynomial.

§ 22. Theorem 8, special discrete measures, l arbitrary.

In Theorem 8 only the three first cases (a), (b), (c) of the special discrete measures defined in § 5.3 occur. (We were not able to prove the assertion of Theorem 8 for $t_n(x)$.) We follow again the notation introduced in § 5.3 and we shall make use of some elementary facts mentioned in § 5 and § 20.

(a) Poisson-Charlier polynomials. The polynomials $Q_n(x) = c_n(x)$ satisfy the normalization $Q_n(0) = 1$. The proof of Theorem 8 is based on the following identity:

$$(22.1)\qquad c_{n+1}(x) - c_n(x) = -\frac{x}{a}c_n(x-1), \qquad n = 0, 1, 2, \ldots .$$

It is derived from the generating function (20.1) by differentiation with respect to z and equating coefficients of z^n.

We generalize (22.1) as follows:

$$(22.2)\quad \Delta^h c_n(x) = (-1)^h\frac{x(x-1)\ldots(x-h+1)}{a^h}c_n(x-h) = (-1)^h\frac{[x]_h}{a^h}c_n(x-h);$$
$$h = 1, 2, \ldots ,$$

where Δ stands for the first difference with respect to n:

$$\Delta a_n = a_{n+1} - a_n ,$$

and we use the notation

$$[x]_h = x(x-1) \dots (x-h+1) \; ; \; [x]_0 = 1 .$$

For the proof we apply the operation Δ to both sides of (22.2) and obtain

$$(\Delta\Delta^h) c_n(x) = \Delta^{h+1} c_n(x) = (-1)^h \frac{[x]_h}{a^h} \cdot \left(-\frac{x-h}{a}\right) c_n(x-h-1)$$

so that induction yields the formula. Now we form the matrix product

$$\left(c_{n+\mu+\nu}(x)\right) \left(\binom{\nu}{\mu}(-1)^{\nu-\mu}\right), \qquad \mu, \nu = 0, 1, \dots, l-1 .$$

Since

$$\sum_{k=0}^{l-1} c_{n+\mu+k}(x) \binom{\nu}{k} (-1)^{\nu-k} = \Delta^\nu c_{n+\mu}(x) = (-1)^\nu \frac{[x]_\nu}{a^\nu} c_{n+\mu}(x-\nu)$$

we have the following identity, valid whether l is even or odd:

$$(22.3) \quad T(c_n(x), c_{n+1}(x), \dots, c_{n+l-1}(x)) = (-1)^{l(l-1)/2} \frac{[x]_1 [x]_2 \dots [x]_{l-1}}{a^{l(l-1)/2}} \cdot [c_{n+\mu}(x-\nu)]_0^{l-1} .$$

In the last determinant μ and ν run from 0 to $l-1$. Hence $T=0$ for $x = 0, 1, \dots, l-2$. Let x be an integer, $x \geqq l-1$. Rearranging the columns of the determinant we obtain, following the notation (1.3), $Q_n(x) = c_n(x)$, that

$$[c_{n+\mu}(x-\nu)]_0^{l-1} = (-1)^{l(l-1)/2} Q \begin{pmatrix} n, & n+1, \dots, n+l-1 \\ x-l+1, & x-l+2, \dots, x \end{pmatrix} .$$

Thus we find from (22.3), in view of Theorem 3, l even,

$$\operatorname{sgn} T = (-1)^{l(l-1)/2} \cdot (-1)^{l(l-1)/2} \cdot (-1)^{l/2} = (-1)^{l/2} .$$

The odd order determinants T (l odd and fixed) with variable n, where each integer selection $x \geqq l$ induces a different sequence in n, constitute a Sturm set. This is an immediate application of Theorem 4′ in § 10.3.

(b) Meixner polynomials, l even. The polynomials

$$Q_n(x) = M_n(\beta, \gamma; x)$$

satisfy the condition $Q_n(0) = 1$. The proof of Theorem 8 is based on the following identity:

$$M_{n+1}(\beta, \gamma; x) - M_n(\beta, \gamma; x) = -\left(\frac{1}{\gamma} - 1\right)\frac{x}{\beta} M_n(\beta+1, \gamma; x-1). \tag{22.4}$$

It can be derived from the generating function (20.2), (20.3) where we assume, for ease of exposition that x is an integer, $x \geq 0$. Then the first part of (5.14) must be used, and of course (22.4) will hold as an identity in x. (Hereafter, in writing M_n the dependence on γ will be suppressed.) From (22.4) we conclude as in (a):

$$\Delta^h M_n(\beta; x) = (-1)^h\left(\frac{1}{\gamma} - 1\right)^h \frac{[x]_h}{(\beta)_h} M_n(\beta+h; x-h). \tag{22.5}$$

Here we use also the symbol

$$(\beta)_h = \beta(\beta+1)\dots(\beta+h-1)\ ;\ (\beta)_0 = 1.$$

As in (a), we form the matrix product

$$\left(M_{n+\mu+\nu}(\beta; x)\right)\left(\binom{\nu}{\mu}(-1)^{\nu-\mu}\right). \tag{22.6}$$

Since

$$\begin{aligned}\sum_{k=0}^{l-1} M_{n+\mu+k}(\beta; x)\binom{\nu}{k}(-1)^{\nu-k} &= \Delta^\nu M_{n+\mu}(\beta; x)\\ &= (-1)^\nu\left(\frac{1}{\gamma} - 1\right)^\nu \frac{[x]_\nu}{(\beta)_\nu} \cdot M_{n+\mu}(\beta+\nu; x-\nu)\end{aligned} \tag{22.7}$$

we obtain the following identity:

$$\begin{aligned}&T(M_n(\beta; x),\ M_{n+1}(\beta; x), \dots, M_{n+l-1}(\beta; x))\\ &= (-1)^{l(l-1)/2}\left(\frac{1}{\gamma} - 1\right)^{l(l-1)/2} \frac{[x]_1[x]_2\dots[x]_{l-1}}{(\beta)_1(\beta)_2\dots(\beta)_{l-1}}\left[M_{n+\mu}(\beta+\nu; x-\nu)\right]_0^{l-1}.\end{aligned} \tag{22.8}$$

Thus again $T = 0$ for $x = 0, 1, \dots, l-2$. Henceforth we consider only integer values of $x, x \geq l-1$. By invoking the symmetry property (5.23) it is convenient to pass to the determinant

$$(22.9)\quad [M_{x-\nu}(\beta+\nu\,;n+\mu)]_0^{l-1} = (-1)^{l(l-1)/2}\,[M_{x-l+1+\nu}(\beta+l-1-\nu\,;n+\mu)]_0^{l-1}$$
$$= (-1)^{l(l-1)/2}\,[M_{x-l+1+\mu}(\beta+l-1-\mu\,;\,n+\nu)]_0^{l-1};$$

here we have first rearranged the columns and then interchanged rows and columns.

Up to this point l may be even or odd.

Now let us assume that l is even. We are prepared to apply the closing remarks of § 8 where n has to be replaced by $x-l+1$. We have to show that the polynomials $M_{x-l+1+\mu}(\beta+l-1-\mu\,;y)$ (in order to avoid confusion we write y for the independent variable) are orthogonal to y^ρ, $\rho<x-l+1$, relative to the measure defined by the following jump at y [cf. (5.24) where β must be replaced be $\beta+l-1$]:

$$(22.10)\qquad j(y) = (1-\gamma)^{\beta+l-1}\,\frac{(\beta+l-1)_y\,\gamma^y}{y!}, \qquad y=0,1,\ldots.$$

Indeed,

$$\sum_{y=0}^{\infty}(1-\gamma)^{\beta+l-1}\,\frac{(\beta+l-1)_y\,\gamma^y}{y!}\,M_{x-l+1+\mu}(\beta+l-1-\mu\,;y)\cdot y^\rho$$
$$=(1-\gamma)^{\mu}\sum_{y=0}^{\infty}(1-\gamma)^{\beta+l-1-\mu}\,\frac{(\beta+l-1-\mu)_y\,\gamma^y}{y!}\cdot M_{x-l+1+\mu}(\beta+l-1-\mu\,;y)$$
$$\cdot\,\frac{(\beta+l-1-\mu+y)_\mu\,y^\rho}{(\beta+l-1-\mu)_\mu}$$

since

$$(a)_y = (a-\mu)_y\,\frac{(a-\mu+y)_\mu}{(a-\mu)_\mu}.$$

Now

$$\frac{(\beta+l-1-\mu+y)_\mu\,y^\rho}{(\beta+l-1-\mu)_\mu}$$

is a polynomial in y of degree

$$\mu+\rho<x-l+1+\mu$$

so that the last sum is zero.

This establishes the assertion of Theorem 8.

(c) Krawtchouk polynomials, l even. The changes are insignificant.

We write (displaying the dependence of the polynomials on the parameter N since N will vary in some of the subsequent formulas):

$$(22.11) \qquad Q_n^{(N)}(x) = \frac{k_n(x)}{k_n(0)}\,; \qquad Q_n^{(N)}(0) = 1\,.$$

We have

$$(22.12) \qquad Q_{n+1}^{(N)}(x) - Q_n^{(N)}(x) = -\frac{x}{Np}\,Q_n^{(N-1)}(x-1)\,, \qquad n \leqq N-1\,,$$

$$(22.13) \qquad \Delta^h Q_n^{(N)}(x) = (-1)^h \frac{[x]_h}{[N]_h p^h}\,Q_n^{(N-h)}(x-h), \qquad n \leqq N-h\,.$$

The second equation arises from the first one in a similar fashion as in (a) and (b). For the proof of (22.12), let x be an integer, $0 \leqq x \leqq N$. Denoting by $f(z)$ the generating function (20.4) we have

$$N \sum_{n=0}^{N-1} \binom{N-1}{n} \left(Q_n^{(N)}(x) - Q_{n+1}^{(N)}(x)\right) z^n = (-1-z)\,f'(z) + Nf(z)\,.$$

Incidentally, this is the "Laguerre derivative" of $f(z)$ with respect to -1 [see 11, Vol. 2, p. 61, p. 246, Problem 128, $\zeta = -1$]. The last expression is

$$= (1+z)^{N-x}\left(1 - \frac{q}{p}z\right)^{x-1} \left\{(x-N)\left(1 - \frac{q}{p}z\right) - \frac{q}{p}\,x(-1-z) + N\left(1 - \frac{q}{p}z\right)\right\}$$

$$= \frac{x}{p}(1+z)^{N-x}\left(1 - \frac{q}{p}z\right)^{x-1} = \frac{x}{p}\sum_{n=0}^{N-1}\binom{N-1}{n} Q_n^{(N-1)}(x-1)\,z^n$$

which yields (22.12). Of course (22.12) and (22.13) are identities in x since they hold for $N+1$ distinct values of x. Hence we have the further identity, $n + 2l - 2 \leqq N$,

$$(22.14) \qquad T\left(Q_n^{(N)}(x)\,,\, Q_{n+1}^{(N)}(x)\,, \ldots ,\, Q_{n+l-1}^{(N)}(x)\right)$$
$$= (-1)^{l(l-1)/2} \frac{[x]_1\,[x]_2 \ldots [x]_{l-1}}{[N]_1\,[N]_2 \ldots [N]_{l-1}\,p^{l(l-1)/2}} \left[Q_{n+\mu}^{(N-\nu)}(x-\nu)\right]_0^{l-1}.$$

Thus $T = 0$ for $x = 0, 1, \ldots, l-2$. Using the symmetry relation (5.27) we see that also $T = 0$ for $x = N, N-1, \ldots, N-l+2$. Let x be an integer, $l-1 \leqq x \leqq N-l+1$. We use the symmetry (5.26) and obtain the determinant

$$[Q_{x-\nu}^{(N-\nu)}(n+\mu)]_0^{l-1} = (-1)^{l(l-1)/2}\,[Q_{x-l+1+\mu}^{(N-l+1+\mu)}(n+\nu)]_0^{l-1}.$$

Up to this point l may be even or odd.

Now let l be even. Again we employ the closing remarks of § 8 replacing there n by $x-l+1$, and show that all polynomials $Q_{x-l+1+\mu}^{(N-l+1+\mu)}(y)$ are orthogonal to y^ρ, $\rho<x-l+1$, relative to the measure defined by the jump at y [(5.28)]:

$$j(y) = \binom{N-l+1}{y} p^y q^{N-l+1+y}, \qquad y = 0, 1, \ldots, N-l+1 .$$

The spectrum thus defined contains all occuring arguments $n+\nu$ since

$$n+l-1 \leqq N-l+1 .$$

Moreover no occuring degree $x-l+1+\mu$ still exceeds $N-l+1$ since

$$x \leqq N-l+1 .$$

For the proof of the orthogonality we note that

$$\sum_{y=0}^{N-l+1} \binom{N-l+1}{y} p^y q^{N-l+1-y} \cdot Q_{x-l+1+\mu}^{(N-l+1+\mu)}(y) \cdot y^\rho$$

$$= q^{-\mu} \sum_{y=0}^{N-l+1+\mu} \binom{N-l+1+\mu}{y} p^y q^{N-l+1+\mu-y} Q_{x-l+1+\mu}^{(N-l+1+\mu)}(y) \cdot \frac{\binom{N-l+1}{y} y^\rho}{\binom{N-l+1+\mu}{y}} .$$

The latter quotient is a polynomial in y of degree $\mu+\rho$ (vanishing for $y>N-l+1$) so that the sum is zero and the assertion is established.

(d) Meixner polynomials, l odd. We return again to the Meixner polynomials and analyze now the oscillation properties of the determinant (22.8) of the Turán type for l odd and x belonging to the spectrum. We consider for each integer x, $x \geqq l-1$, the quantity (22.8) as a sequence in the variable n which we denote by the symbol $\psi_x(l, \beta; n) = \psi_x(n)$. We will also find it convenient to construct the linear interpolation $\psi_x(y)$ for $y \geqq 0$.

Theorem 8': Let l be a fixed odd integer and x an integer, $x \geqq l-1$. Then $\psi_x(y)$ has for $y \geqq 0$ exactly $x-l+1$

n o d a l z e r o s. In particular, the sequence (22.8), $n=0, 1, 2, \ldots$, has precisely $x-l+1$ sign changes.

We note that the interlacing property is missing so that no Sturmian set appears. Still the argument will follow very closely the standard argument of § 2.

We first establish the signs of the following quantities:

(22.15) (i): $\psi_{l-1}(n)$, $n=0, 1, 2, \ldots$; (ii): $\psi_x(0)$, $x \geqq l-1$;
(iii): $\psi_x(n)$ for large n, $x \geqq l-1$.

Denoting the second matrix in (22.6) by H we form the matrix

$$H'TH = H' \cdot (M_{n+\mu+\nu}(\beta\,;x)) \cdot H. \tag{22.16}$$

In view of (22.7) we obtain for the general element of (22.16):

$$\begin{aligned}
&(-1)^\nu \left(\frac{1}{\gamma}-1\right)^\nu \frac{[x]_\nu}{(\beta)_\nu} \sum_{k=0}^{\mu} \binom{\mu}{k} (-1)^{\mu-k} M_{n+k}(\beta+\nu\,;x-\nu) \\
&= (-1)^\nu \left(\frac{1}{\gamma}-1\right)^\nu \frac{[x]_\nu}{(\beta)_\nu} \Delta^\mu M_n(\beta+\nu\,;x-\nu) \\
&= (-1)^{\mu+\nu} \left(\frac{1}{\gamma}-1\right)^{\mu+\nu} \frac{[x]_\nu}{(\beta)_\nu} \frac{[x-\nu]_\mu}{(\beta+\nu)_\mu} \cdot M_n(\beta+\mu+\nu\,;x-\mu-\nu) \\
&= (-1)^{\mu+\nu} \left(\frac{1}{\gamma}-1\right)^{\mu+\nu} \frac{[x]_{\mu+\nu}}{(\beta)_{\mu+\nu}} \cdot M_n(\beta+\mu+\nu\,;x-\mu-\nu).
\end{aligned} \tag{22.17}$$

Here we used (22.5). Hence we obtain the following alternative form of (22.8):

$$\begin{aligned}
\psi_x(n) &= T(M_n(\beta\,;x), M_{n+1}(\beta\,;x), \ldots, M_{n+l-1}(\beta\,;x)) \\
&= \left(\frac{1}{\gamma}-1\right)^{l(l-1)} \left[\frac{[x]_{\mu+\nu}}{(\beta)_{\mu+\nu}} M_n(\beta+\mu+\nu\,;x-\mu-\nu)\right]_0^{l-1}.
\end{aligned} \tag{22.18}$$

Now we proceed to the evaluation of the quantities (22.15).

(i): For $x=l-1$ all elements of (22.18) are zero if

$$x-\mu-\nu+1 = l-\mu-\nu \leqq 0, \quad l \leqq \mu+\nu.$$

Thus all elements under the diagonal $\mu+\nu=l-1$ vanish so that since $M_n(\beta\,;0)=1$,

$$\operatorname{sgn} \psi_{l-1}(n) = (-1)^{(l-1)/2}. \tag{22.19}$$

(ii): We use again (22.18) so that since $M_0(\beta\,;\,x)=1$,

$$\operatorname{sgn} \psi_x(0) = \operatorname{sgn}\left[\frac{[x]_{\mu+\nu}}{(\beta)_{\mu+\nu}}\right]_0^{l-1}.$$

This determinant is

$$= \prod_{\mu=0}^{l-1} \frac{[x]_\mu}{(\beta)_\mu} \cdot \left[\frac{[x-\mu]_\nu}{(\beta+\mu)_\nu}\right]_0^{l-1}$$

where the first factor is positive. In the second factor we replace μ by a variable t and decompose in partial fractions as follows:

$$\frac{[x-t]_\nu}{(\beta+t)_\nu} = \rho_{0\nu} + \frac{\rho_{1\nu}}{\beta+t} + \frac{\rho_{2\nu}}{\beta+t+1} + \ldots + \frac{\rho_{\nu\nu}}{\beta+t+\nu-1}\,;$$

$$\rho_{00} = 1\;,\; \rho_{\nu\nu} = (-1)^{\nu-1}\frac{[x+\beta+\nu-1]_\nu}{(\nu-1)\,!}\,, \qquad \nu \geqq 1\,.$$

Writing $\rho_{\mu\nu}=0$ for $\mu>\nu$ we form the matrix product

$$\left(\frac{1}{\beta+\mu+\nu-1}\right)(\rho_{\mu\nu})$$

where $\beta+\mu+\nu-1$ must be replaced by 1 for $\nu=0$. Now

$$\sum_{k=0}^{\nu} \frac{\rho_{k\nu}}{\beta+\mu+k-1} = \frac{[x-\mu]_\nu}{(\beta+\mu)_\nu}$$

so that

$$\operatorname{sgn}\psi_x(0) = \operatorname{sgn}\prod_{\nu=0}^{l-1}\rho_{\nu\nu}\cdot\operatorname{sgn}\left[\frac{1}{\beta+\mu+\nu-1}\right]_0^{l-1} \tag{22.20}$$

$$= (-1)^{(l-1)(l-2)/2} = (-1)^{(l-1)/2}.$$

As to the last determinant, cf. for instance [11, Vol. 2, p. 98, Problem 3].

(iii): From (20.2), (20.3) we conclude the following asymptotic formula valid for integer x, $x=0,1,2,\ldots$,

$$M_n(\beta\,;\,x) \cong (-1)^x \frac{\left(\frac{1}{\gamma}-1\right)^x}{(\beta)_x}\, n^x \quad \text{as} \quad n\to\infty\,. \tag{22.21}$$

Thus those elements of (22.18) for which $x-\mu-\nu \geqq 0$, are [cf. (22.17)]

$$\cong (-1)^{\mu+\nu}\left(\frac{1}{\gamma}-1\right)^{\mu+\nu}\frac{[x]_{\mu+\nu}}{(\beta)_{\mu+\nu}}\cdot(-1)^{x-\mu-\nu}\frac{\left(\frac{1}{\gamma}-1\right)^{x-\mu-\nu}}{(\beta+\mu+\nu)_{x-\mu-\nu}}n^{x-\mu-\nu}$$

$$\cong (-1)^{x}\left(\frac{1}{\gamma}-1\right)^{x}\frac{[x]_{\mu+\nu}}{(\beta)_{x}}n^{x-\mu-\nu}$$

so that for large n

$$\operatorname{sgn}\psi_x(n) = (-1)^{xl}\operatorname{sgn}\left[\,[x]_{\mu+\nu}\right]_0^{l-1}.$$

In the μ^{th} row of the last determinant we take out the factor $[x]_\mu$ so that the remaining elements in the μ^{th} row will be

$$1\,,\ x-\mu\,,\ (x-\mu)(x-\mu-1)\,,\ (x-\mu)(x-\mu-1)(x-\mu-2)\,,\ \dots$$

which can be reduced, after appropriate combination of the columns to

$$1\,,\ -\mu\,,\ (-\mu)^2\,,\ (-\mu)^3\,,\ \dots\,.$$

This Vandermondian has the sign $(-1)^{l(l-1)/2}$ so that for large n

(22.22) $$\operatorname{sgn}\psi_x(n) = (-1)^{xl+l(l-1)/2} = (-1)^{x+(l-1)/2}.$$

Now having established the signs of the quantities (22.15) we apply the Sylvester identity to the determinant on the right of (22.8) (replacing $l-1$ by l) which is positive if its order is even. We have

$$[M_{n+\mu}(\beta+\nu\,;\,x-\nu)]_0^l\cdot[M_{n+1+\mu}(\beta+1+\nu\,;\,x-1-\nu)]_0^{l-2}$$
$$=\begin{vmatrix}[M_{n+\mu}(\beta+\nu\,;\,x-\nu)]_0^{l-1} & [M_{n+\mu}(\beta+1+\nu\,;\,x-1-\nu)]_0^{l-1}\\ [M_{n+1+\mu}(\beta+\nu\,;\,x-\nu)]_0^{l-1} & [M_{n+1+\mu}(\beta+1+\nu\,;\,x-1-\nu)]_0^{l-1}\end{vmatrix}.$$

In the present case l is odd so that both determinants on the left are positive. Taking (22.8) into account (this identity is valid whether l is even or odd) we obtain the following important inequality:

(22.23) $$\begin{vmatrix}\psi_x(l\,,\,\beta\,;\,n) & \psi_{x-1}(l\,,\,\beta+1\,;\,n)\\ \psi_x(l\,,\,\beta\,;\,n+1) & \psi_{x-1}(l\,,\,\beta+1\,;\,n+1)\end{vmatrix} > 0\,.$$

As an immediate consequence we point out that the two elements in the first column can not vanish simultaneously so that $\psi_x(y)$ possesses only nodal zeros.

The remaining argument is based on induction with respect to x. We shall not repeat the details which run parallel to those of § 9.

Let $x-1 \geqq l-1$. We assume that $\psi_{x-1}(y)$ has exactly

$$x-1-l+1 = x-l$$

nodal zeros for each choice of the parameter β, $\beta > 0$. Between each two successive zeros of $\psi_{x-1}(l, \beta+1; y)$ we infer the existence of a nodal zero of $\psi_x(l, \beta; y)$, and conversely.

Further we make the usual observation about the extreme zeros. Let y_1 be the first zero of $\psi_{x-1}(l, \beta+1; y)$ located say in the interval $n \leqq y < n+1$. We know on the basis of (22.20) that

$$(-1)^{(l-1)/2}\, \psi_{x-1}(l, \beta+1; n) \geqq 0$$

and

$$(-1)^{(l-1)/2}\, \psi_{x-1}(l, \beta+1; n+1) < 0 .$$

It follows easily from (22.23) that

$$(-1)^{(l-1)/2}\, \psi_x(l, \beta; y_1) < 0 .$$

Since

$$(-1)^{(l-1)/2}\, \psi_x(l, \beta; 0) > 0$$

we infer the existence of a nodal zero for $\psi_x(l, \beta; y)$ lying in the interval $(0, y_1)$. Similarly, examination of the sign (22.22) in conjunction with (22.23) yields the existence of a nodal zero of $\psi_x(l, \beta; y)$ which exceeds in magnitude the largest zero of $\psi_{x-1}(l, \beta+1; y)$.

This finishes the proof of Theorem 8′.

As to the Krawtchouk polynomials, by a similar method we can prove the following

Theorem 8″: *Let l be odd. We use the notation in (c) and consider in particular the determinant (22.14) of the Turán type which we denote also by $\psi_x(n)$, its linear interpolation by $\psi_x(y)$. If x is an integer,*

$$l-1 \leqq x \leqq N-l+1 ,$$

the function $\psi_x(y)$ will have exactly $x-l+1$ nodal zeros all located in the interval $(0, N-2l+2)$.

Chapter 4. AUGMENTED DETERMINANTS OF THE WRONSKI AND TURAN TYPE

§ 23. Theorem 9, classical polynomials, Wronski type, $l = 2$.

1. In this section we investigate the nature of the zeros of the augmented polynomial system $\varphi_n(k\,;x) = \varphi_n(x)$ defined by (1.19) where $\{Q_n(x)\}$ represents one of the systems of classical orthogonal polynomials defined in Theorem 5. Pertaining to the zeros of $\varphi_n(x)$ the normalization imposed on $Q_n(x)$ is clearly irrelevant. The normalization to be used below will be partly different from that of Theorem 5; it would rather follow the normalization of Theorem 1.

The proof of Theorem 9 will be based on the differential equation of these polynomials which has in all cases the following form [(5.5), (5.13) and 13, (5.5.2)]:

$$r(x)\,Q''_n(x) + s(x)\,Q'_n(x) + \lambda_n Q_n(x) = 0, \qquad n = 0, 1, 2, \dots, \tag{23.1}$$

where $r(x)$ and $s(x)$ are analytic in the interior and in the finite boundary points of the interval of orthogonality; $r(x)$ is positive in the interior of the same interval (spectrum). The eigenvalues λ_n are strictly increasing and positive. The emphasis of the differential equation is important because it suggests certain generalizations of Theorem 9 to determinants whose elements are successive eigenfunctions of certain Sturm-Liouville problems. Such extensions will be dealt with elsewhere.

2. For the sake of definiteness we assume that

$$Q_n(x) = k_n(-x)^n + \dots, \quad k_n > 0, \quad \text{i.e.} \quad Q_n(-\infty) = +\infty\,.$$

This is the case for the Laguerre polynomials; $P_n^{(\lambda)}(-x)$ and $H_n(-x)$ satisfy also this condition. According to Theorem 1 the polynomial $\varphi_{k+1}(x)$ will be negative for all real x. Since $Q_n(x)/Q_{n+1}(x)$ is strictly monotonic exterior to the spectrum [cf. 13, (3.3.9)], we easily deduce that $\varphi_n(x)$ never vanishes exterior to the spectrum.

Let x_0 be on the spectrum; two consecutive polynomials, say $\varphi_n(x)$ and $\varphi_{n+1}(x)$ can not vanish for $x = x_0$ simultaneously; this is a consequence of the following simple algebraic remark which is actually the first step in verifying the Sturm properties of the sequence

$$\{\varphi_n(x)\,;\, n = k+1\,,\, k+2\,, \ldots\}\,.$$

We simplify the notation by dropping the argument $x = x_0$. The equations

$$\begin{aligned} Q_k\, Q'_{n-1} - Q'_k\, Q_{n-1} - \varphi_{n-1} &= 0\,, \\ Q_k\, Q'_n - Q'_k\, Q_n - \varphi_n &= 0\,, \\ Q_k\, Q'_{n+1} - Q'_k\, Q_{n+1} - \varphi_{n+1} &= 0 \end{aligned}$$

are homogeneous and linear in the quantities Q_k , $-Q'_k$, -1 so that

$$\begin{vmatrix} Q'_{n-1} & Q_{n-1} & \varphi_{n-1} \\ Q'_n & Q_n & \varphi_n \\ Q'_{n+1} & Q_{n+1} & \varphi_{n+1} \end{vmatrix} = 0\,.$$

Thus if $\varphi_n = 0$ for $x = x_0$, we have

$$\varphi_{n-1} \begin{vmatrix} Q'_n & Q_n \\ Q'_{n+1} & Q_{n+1} \end{vmatrix} + \varphi_{n+1} \begin{vmatrix} Q'_{n-1} & Q_{n-1} \\ Q'_n & Q_n \end{vmatrix} = 0\,.$$

Theorem 1 implies that both determinants are positive so that if

$$\varphi_n = \varphi_{n+1} = 0$$

we had also $\varphi_{n-1} = 0$. Further application of this remark would lead to $\varphi_{k+1} = 0$ which is a contradiction. Thus if $\varphi_n(x_0) = 0$ we must have

$$\varphi_{n-1}(x_0)\, \varphi_{n+1}(x_0) < 0\,. \tag{23.2}$$

For the same $x = x_0$ we show now that either

$$\varphi'_n(x_0)\, \varphi_{n-1}(x_0) < 0 \quad \text{or} \tag{23.3}$$

$$\varphi'''_n(x_0)\, \varphi_{n-1}(x_0) < 0\,, \tag{23.4}$$

the second inequality holding when $\varphi_n(x_0) = \varphi'_n(x_0) = 0$ in which case also $\varphi''_n(x_0) = 0$.

Executing the required differentiation, using $\varphi_n(x_0) = 0$ and the differential equation (23.1), we find (see below (23.10))

$$r(x_0)\, \varphi'_n(x_0) = (\lambda_k - \lambda_n)\, Q_k(x_0)\, Q_n(x_0)\,. \tag{23.5}$$

If also $Q_k(x_0) = 0$, then $Q'_k(x_0) \neq 0$ and thus $Q_n(x_0) = 0$ [9]. Similarly, if

9. Two of the classical polynomials belonging to the same system might vanish simultaneously. Trivially, in the case of the ultraspherical polynomials when k and n are both odd, then $Q_k(0) = Q_n(0) = 0$.

$Q_n(x_0) = 0$, then $Q'_n(x_0) \neq 0$ and again $Q_k(x_0) = 0$. In this event we obtain $\varphi'_n(x_0) = \varphi''_n(x_0) = 0$ but

$$r(x_0)\,\varphi'''_n(x_0) = 2(\lambda_k - \lambda_n)\,Q'_k(x_0)\,Q'_n(x_0) \tag{23.6}$$

and the right hand side is not zero.

Thus, x_0 being a zero of $\varphi_n(x)$ we find that two cases are possible: (a) x_0 is a simple zero in which case $Q_k(x_0) \neq 0$, $Q_n(x_0) \neq 0$; (b) x_0 is exactly of order three in which case

$$Q_k(x_0) = Q_n(x_0) = 0\,.$$

In order to prove (23.3) or (23.4) we derive an alternative expression for $\varphi_{n-1}(x_0)$, when $\varphi_n(x_0) = 0$, as follows. Observe first that $\varphi_{n-1}(x_0) \neq 0$ as pointed out already. In case x_0 is a simple zero of $\varphi_n(x)$ we have

$$\varphi_{n-1}(x_0) = -\frac{Q_k(x_0)}{Q_n(x_0)} \begin{vmatrix} Q_{n-1}(x_0) & Q_n(x_0) \\ Q'_{n-1}(x_0) & Q'_n(x_0) \end{vmatrix}\,.$$

Thus

$$\begin{aligned} &r(x_0)\,\varphi'_n(x_0)\,\varphi_{n-1}(x_0) \\ &= -(\lambda_k - \lambda_n)\,[Q_k(x_0)]^2 \begin{vmatrix} Q_{n-1}(x_0) & Q_n(x_0) \\ Q'_{n-1}(x_0) & Q'_n(x_0) \end{vmatrix} < 0\,. \end{aligned} \tag{23.7}$$

If x_0 is a zero of $\varphi_n(x)$ of order three (the other possibility) then

$$\varphi_{n-1}(x_0) = -Q_{n-1}(x_0)\,Q'_k(x_0)$$

so that

$$\begin{aligned} &r(x_0)\,\varphi'''_n(x_0)\,\varphi_{n-1}(x_0) \\ &= -2(\lambda_k - \lambda_n)\,[Q'_k(x_0)]^2 \begin{vmatrix} Q_{n-1}(x_0) & Q_n(x_0) \\ Q'_{n-1}(x_0) & Q'_n(x_0) \end{vmatrix} < 0\,. \end{aligned} \tag{23.8}$$

The two latter inequalities yield (23.3), (23.4); with their aid and by (23.2) we may now establish the Sturmian properties asserted in Theorem 9. We proceed as in § 2 although

$$\varphi_n(x_0) = \varphi'_n(x_0) = \varphi''_n(x_0) = 0$$

is a possibility; in this case (23.4) [instead of (23.3)] must be used. Since $\varphi_n(x)$ is never zero outside the spectrum we need only the following further information:

(23.9) $$\operatorname{sgn} \varphi_n(-\infty) = -1\,, \quad \operatorname{sgn} \varphi_n(+\infty) = (-1)^{n+k}\,.$$

This establishes the proof.

It is a trivial matter to construct orthogonal polynomial systems such that $\varphi_{k+2}(x)$ has three real zeros so that Theorem 9 is not valid generally. We leave this to the interested reader.

3. With the aid of (23.1) we prove now that the system $\{\varphi_n(x)\}$ constitutes a complete set of eigenfunctions of a second order differential operator in the same sense as the classical polynomials are eigenfunctions of such an operator. The coefficients of this operator may depend now on k.

First, by definition we have (dropping the arguments for the sake of brevity)

$$\varphi_n = -Q'_k Q_n + Q_k Q'_n$$

and by (23.1)

(23.10) $$r\varphi'_n + s\varphi_n = (\lambda_k - \lambda_n)\, Q_k Q_n\,.$$

Solving for Q_n and Q'_n we obtain

(23.11) $$Q_n = \frac{r\varphi'_n + s\varphi_n}{(\lambda_k - \lambda_n)\, Q_k}$$

and

(23.12) $$Q'_n = \frac{(r\varphi'_n + s\varphi_n)\, Q'_k}{(\lambda_k - \lambda_n)\, Q_k^2} + \frac{\varphi_n}{Q_k}\,.$$

Differentiating (23.10) and using (23.11) and (23.12) we produce the differential equation which exhibits φ_n as eigenfunctions of a differential operator:

(23.13) $$(r\varphi'_n + s\varphi_n)' - \frac{2Q'_k}{Q_k}(r\varphi'_n + s\varphi_n) = (\lambda_k - \lambda_n)\, \varphi_n\,,$$

$$n = 0, 1, 2, \ldots\,.$$

In deriving this we made use solely of (23.1) so that any function φ_n defined by (1.19) will satisfy (23.13) provided the involved functions Q_n (not necessarily polynomials) satisfy the differential equation (23.1).

It should be also observed that the coefficient of φ'_n in (23.13) has singularities interior to the spectrum, namely at the zeros of Q_k and, in this respect only, the differential equation is not of the standard Sturm-Liouville type.

4. Finally we deal with a converse proposition restricting ourselves, for the sake of brevity, to the cases of the ultraspherical and Laguerre polynomials. The necessary modifications in the remaining case of the Hermite polynomials will be slight.

Let γ be a constant, k an integer. We prove that the only analytical solution f of

$$(rf'+sf)'-\frac{2Q'_k}{Q_k}(rf'+sf)=\gamma f \tag{23.14}$$

which satisfies certain regularity conditions on the spectrum (see below), is a multiple of φ_n for some $n=0, 1, 2, \ldots,$ $\gamma=\lambda_k-\lambda_n$. To this end, we observe that analyticity of f requires

$$rf'+sf=Q_k g \tag{23.15}$$

where g is again analytic on the spectrum. Inserting this expression into (23.14) and performing the indicated operations we obtain

$$Q_k g'-Q'_k g=\gamma f. \tag{23.16}$$

The latter equation yields

$$\begin{aligned}\gamma(rf'+sf)&=Q_k(rg''+sg')-g(rQ''_k+sQ'_k)\\&=Q_k(rg''+sg')+g\lambda_k Q_k;\end{aligned} \tag{23.17}$$

in the last step we used (23.1). Combining (23.15) and (23.17) shows that g satisfies

$$rg''+sg'=(\gamma-\lambda_k)g. \tag{23.18}$$

Now we proceed to the discussion of the two cases mentioned above.

(a) Ultraspherical polynomials:

$$r(x)=1-x^2\ ,\ s(x)=-(2\lambda+1)x\ ,\ \lambda_n=n(n+2\lambda)\ ,\ \lambda>-\frac{1}{2}.$$

We assume [in addition to (23.14)] that $f(x)$ is analytic in the

closed interval $-1 \leqq x \leqq 1$, and we conclude that $f = \text{const.}\ \varphi_n$ for some n.

It is well known [cf. 13, 4.2, 4.21, (4.7.6)] that the only solution of (23.18) regular at $x = 1$, is the hypergeometric function

$$g(x) = F\left(-a\,,\, a + 2\lambda\,;\, \lambda + \frac{1}{2}\,;\, \frac{1-x}{2}\right).$$

If $a \neq n$ or $-n - 2\lambda$, $n = 0, 1, 2, \ldots$, then all coefficients of this series F will be different from zero and asymptotically of the order $m^{\lambda - 3/2}$ where $m \to \infty$. Consequently this function or its first derivative tends to ∞ as $x \to -1 + 0$.

Thus $g = F$ must be a polynomial and it is easy to show [13, p. 62] that it must be an ultraspherical polynomial. Hence in view of (5.5), $\gamma = k(k + 2\lambda) - n(n + 2\lambda)$, $g = \text{const.}\ Q_n$ for some n and assuming $\gamma \neq 0$, (23.16) reveals that f is, except for a constant factor, one of the φ_n.

The argument is slightly altered for $\gamma = 0$, $g = \text{const.}\ Q_k$. Indeed, we have then from (23.15)

$$\frac{d}{dx}\{(1-x^2)^{\lambda+1/2} f(x)\} = \text{const.}\ (1-x^2)^{\lambda-1/2}\,[Q_k(x)]^2$$

so that $(1-x^2)^{\lambda+1/2} f(x)$ must be monotonic in $-1 < x < 1$; this function tends to 0 if $x \to \pm 1$, hence $f(x) \equiv 0$.

(b) Laguerre polynomials:

$$r(x) = x\ ,\quad s(x) = \alpha + 1 - x\ ,\quad \lambda_n = n\ ,\quad \alpha > -1\,.$$

We assume now [in addition to (23.14)] that $f(x)$ is analytic for $x \geqq 0$ and satisfies at $x = +\infty$ the growth condition: $\lim e^{-x}\, x^{h-k+1} f^{(h)}(x) = 0$ for all $h = 0, 1, 2, \ldots$. We conclude for $\gamma \neq 0$ that $f = \text{const.}\ \varphi_n$. For $\gamma = 0$ the conclusion $f \equiv 0$ holds provided the single condition $\lim e^{-x} x^{\alpha+1} f(x) = 0$, $x \to +\infty$, is satisfied.

It is well known [cf. 13, p. 102] that the only solution of (23.8) regular at $x = 0$ is the confluent hypergeometric function

$$g = {}_1F_1(-a\,;\, \alpha + 1\,;\, x)\,.$$

If $a \neq n$, $n = 0, 1, 2, \ldots$, all coefficients of this series will be different from zero and of the asymptotic order $m^{-a-\alpha-1}(m!)^{-1}$ where $m \to \infty$. We show by induction with respect to h that

$$\lim e^{-x} x^h g^{(h)}(x) = 0 \quad \text{as} \quad x \to +\infty$$

which will yield a contradiction.

Indeed from (23.15)

$$e^{-x} g(x) = O(1) \cdot \frac{e^{-x} x |f'(x)| + e^{-x} x |f(x)|}{x^k} \to 0, \quad x \to +\infty .$$

Differentiating (23.15) h times we find

$$x f^{(h+1)} + h f^{(h)} + (\alpha+1-x) f^{(h)} - h f^{(h-1)} = Q_k g^{(h)} + h Q'_k g^{(h-1)} + \ldots .$$

We multiply this equation by $e^{-x} x^{h-k}$; all terms on the right, except possibly the first one, will go to zero, by the induction hypothesis. The same holds for the expression on the left, hence also for the first term on the right. This makes the induction complete.

Thus $g = {}_1F_1$ must be a polynomial and so [cf. 13, p. 100] $\gamma = k - n$, $g = \text{const.}\ Q_n$, $f = \text{const.}\ \varphi_n$, provided $\gamma \neq 0$.

In the case $\gamma = 0$ we obtain as above $g = \text{const.}\ Q_k$ and

$$\frac{d}{dx}[e^{-x} x^{\alpha+1} f(x)] = \text{const.}\ e^{-x} x^{\alpha} [Q_k(x)]^2 ;$$

hence $e^{-x} x^{\alpha+1} f(x)$ must be monotonic, and under the assumption mentioned, $f \equiv 0$.

Another argument based on the completeness property of the Laguerre polynomials can be given requiring slightly weaker conditions on f than indicated in (b).

§ 24. Theorem 10, classical polynomials, Turán type, $l = 2$.

In the same vein of analysis as in the preceding section, we investigate the location of the zeros of the determinants

$$\psi_n(k; x) = \psi_n(x)$$

defined by (1.20) where $\{Q_n(x)\}$ represents again one of the three classical polynomial systems defined and normalized as in Theorem 5. The normalization is now of importance.

1. In all three cases of classical polynomials a differential equation holds which we write now in the slightly altered form

$$[r(x)\, Q'_n(x)]' = -\lambda_n\, \rho(x)\, Q_n(x), \qquad n = 0, 1, 2, \ldots, \tag{24.1}$$

where the $Q_n(x)$ are orthogonal in a certain interval relative to the weight function $\rho(x)$; both $r(x)$ and $\rho(x)$ are analytic and positive in the interior of that interval (spectrum).

Moreover we have in all three cases a recurrence differential relation of the form

$$r(x)\, Q'_n(x) = \mu_n [Q_{n-1}(x) + c(x)\, Q_n(x)]\, \rho(x), \qquad n = 0, 1, 2, \ldots;\ Q_{-1} = 0, \tag{24.2}$$

where $r(x)$ and $\rho(x)$ are the same functions as in (24.1) and $c(x)$ is a suitable polynomial. The constants μ_n are real, $\mu_n \neq 0$.

These two relations play a fundamental role in our proof. It is instructive to describe the three special cases more specifically.

(a) Ultraspherical polynomials:

$$Q_n(x) = P_n^{(\lambda)}(x) / P_n^{(\lambda)}(1), \quad \lambda > -\frac{1}{2}; \quad a = -1, \quad b = +1, \quad Q_n(1) = 1;$$

moreover [(5.5), (5.6)]

$$r(x) = (1-x^2)^{\lambda+1/2}, \quad \rho(x) = (1-x^2)^{\lambda-1/2},$$
$$\lambda_n = n(n+2\lambda), \quad \mu_n = n, \quad c(x) = -x.$$

(b) Laguerre polynomials:

$$Q_n(x) = L_n^{(\alpha)}(x) / L_n^{(\alpha)}(0), \quad \alpha > -1; \quad a = 0, \quad b = +\infty, \quad Q_n(0) = 1;$$

moreover [(5.13), (5.14)]

$$r(x) = e^{-x} x^{\alpha+1}, \quad \rho(x) = e^{-x} x^{\alpha}, \quad \lambda_n = n, \quad \mu_n = -n, \quad c(x) = -1.$$

(c) Hermite polynomials:

$$Q_n(x) = H_n(x), \quad a = -\infty, \quad b = +\infty$$

[(5.15), cf. also 13, (5.5.2)],

$$r(x) = \rho(x) = e^{-x^2}, \quad \lambda_n = \mu_n = 2n, \quad c(x) = 0.$$

In what follows we shall make use of the special Turán inequality $\psi_{k+1}(x) < 0$ where x ranges over the interval of orthogonality. We verify the following elementary facts:

$$(24.3)\quad \begin{cases} \psi_n(-x) = (-1)^{n+k+1}\,\psi_n(x) & \text{in the cases (a) and (c);} \\ \psi_n(1) = 0\ ,\ \psi'_n(1) = \dfrac{2(n-k)}{2\lambda+1} & \text{in the case (a);} \\ \psi_n(0) = \psi'_n(0) = 0\ ,\ \psi''_n(0) = -\dfrac{2(n-k)}{(\alpha+1)^2(\alpha+2)} & \text{in the case (b).} \end{cases}$$

Also we have for large x [(5.11), (5.12); 13, (5.5.4)]

$$(24.4)\quad \begin{cases} \psi_n(x) = (-1)^{n-k}\dfrac{[\Gamma(\alpha+1)]^2(n-k)}{\Gamma(k+\alpha+2)\,\Gamma(n+\alpha+2)}\,x^{n+k+1}+\dots & \text{in the case (b),} \\ \psi_n(x) = -\,2^{n+k}(n-k)\,x^{n+k-1}+\dots & \text{in the case (c).} \end{cases}$$

2. First we point out that if $x = x_0$ is a zero of $\psi_n(x)$ on the spectrum, we have

$$(24.5)\qquad \psi_{n-1}(x_0)\,\psi_{n+1}(x_0) < 0\,.$$

Thus two consecutive ψ_n can never vanish simultaneously. The proof of (24.5) follows the same line as that of (23.2) so that it can be omitted. Next we show that if $\psi_n(x_0) = 0$ we have

$$(24.6)\qquad \begin{cases} \psi'_n(x_0)\,\psi_{n-1}(x_0) > 0\,, \quad \psi'_n(x_0)\,\psi_{n+1}(x_0) < 0 & \text{in (a), (c),} \\ \psi'_n(x_0)\,\psi_{n-1}(x_0) < 0\,, \quad \psi'_n(x_0)\,\psi_{n+1}(x_0) > 0 & \text{in (b).} \end{cases}$$

Since $\psi_n(x_0) = 0$, there exist two real constants γ, δ not both zero such that

$$\gamma\,Q_k(x_0) + \delta\,Q_n(x_0) = 0\,,$$
$$\gamma\,Q_{k+1}(x_0) + \delta\,Q_{n+1}(x_0) = 0\,.$$

If one of these numbers, say δ would be zero, then $Q_k(x_0) = 0$ and $Q_{k+1}(x_0) = 0$ would follow which is impossible. Hence both γ and δ are different from zero so that

$$(24.7)\qquad Q_n(x_0) = t\,Q_k(x_0)\ ,\ Q_{n+1}(x_0) = t\,Q_{k+1}(x_0)\ ,\ t \neq 0\,.$$

Writing $\psi_n(x)$ with the aid of (24.2) as

$$\psi_n(x) = \psi_n = \begin{vmatrix} \frac{r}{\rho} \frac{Q'_{k+1}}{\mu_{k+1}} & \frac{r}{\rho} \frac{Q'_{n+1}}{\mu_{n+1}} \\ Q_{k+1} & Q_{n+1} \end{vmatrix},$$

then differentiating and inserting $x = x_0$ we obtain, making use of (24.1), (24.2) and $\psi_n(x_0) = 0$,

$$\psi'_n(x_0) = \psi'_n = \begin{vmatrix} -\frac{\lambda_{k+1}}{\mu_{k+1}} Q_{k+1} & -\frac{\lambda_{n+1}}{\mu_{n+1}} Q_{n+1} \\ Q_{k+1} & Q_{n+1} \end{vmatrix} + \begin{vmatrix} \frac{r}{\rho} \frac{Q'_{k+1}}{\mu_{k+1}} & \frac{r}{\rho} \frac{Q'_{n+1}}{\mu_{n+1}} \\ Q'_{k+1} & Q'_{n+1} \end{vmatrix}$$

$$= \left(\frac{\lambda_{n+1}}{\mu_{n+1}} - \frac{\lambda_{k+1}}{\mu_{k+1}}\right) Q_{k+1} Q_{n+1} + \frac{\rho}{r} (\mu_{n+1} - \mu_{k+1}) (Q_k + c\, Q_{k+1}) (Q_n + c\, Q_{n+1})$$

$$= t \left\{ \left(\frac{\lambda_{n+1}}{\mu_{n+1}} - \frac{\lambda_{k+1}}{\mu_{k+1}}\right) Q^2_{k+1} + \frac{\rho}{r} (\mu_{n+1} - \mu_{k+1}) (Q_k + c\, Q_{k+1})^2 \right\}.$$

In the latter formulas $x = x_0$. In the last step we made use of (24.7).

In view of the special information given in **1** we conclude that $\frac{\lambda_n}{\mu_n}$ and μ_n are strictly increasing in the case (a); $\frac{\lambda_n}{\mu_n}$ is constant in the cases (b), (c) and μ_n is decreasing in the case (b), increasing in the case (c). Hence the expression in the last curly brackets is positive in the cases (a) (c) and negative in the case (b). Moreover, by appealing to (24.7) we obtain

$$\psi_{n-1}(x_0) = t^{-1} \begin{vmatrix} Q_n(x_0) & Q_{n-1}(x_0) \\ Q_{n+1}(x_0) & Q_n(x_0) \end{vmatrix}.$$

Invoking Turán's inequality establishes the claim of (24.6).

The properties of the zeros of ψ_n as stated in Theorem 10 can now be deduced in a routine manner by using (24.6) as well as the special information (24.3) and (24.4). The details are similar to those of the preceding sections and can be omitted. The proof is thus finished.

3. We continue by deriving a second order differential equation for $\psi_n(x)$ analogous to the equation (23.13) satisfied by $\varphi_n(x)$. In contrast to (23.13) however there does not exist now a unified differential equation satisfied by $\psi_n(x)$ in all the three classical cases which underlie the

definition of $\psi_n(x)$. We discuss each case in a special manner, preferably in the order (c), (b), (a). [10]

(c) This is the simplest case: $Q_n(x) = H_n(x)$. In view of (5.15), we have $\psi_n(x) = -\varphi_n(x)$. Thus the differential equation satisfied by $\psi_n(x)$ in the case of the Hermite polynomials is identical with (23.13).

(b) Next we deal with the case

$$Q_n(x) = L_n^{(\alpha)}(x) \,/\, L_n^{(\alpha)}(0)\,.$$

We recall the familiar identities (5.14), (5.13) and we write the second one in the form

$$[e^{-x}x^{\alpha+1}Q'_n(x)]' = -n\,e^{-x}x^{\alpha}Q_n(x)\,.$$

We obtain

$$(24.8)\qquad \psi = \psi_n(x) = \begin{vmatrix} Q_k(x) & Q_n(x) \\ Q_{k+1}(x) & Q_{n+1}(x) \end{vmatrix} = -x\left(\frac{Q_{n+1}(x)\,Q'_{k+1}(x)}{k+1} - \frac{Q_{k+1}(x)\,Q'_{n+1}(x)}{n+1}\right)$$

and

$$(24.9)\qquad (e^{-x}x^{\alpha}\psi)' = -e^{-x}x^{\alpha+1}\frac{n-k}{(k+1)(n+1)}\,Q'_{k+1}(x)\,Q'_{n+1}(x)\,.$$

Another differentiation and subsequent elimination of $Q_{n+1}(x)$ and $Q'_{n+1}(x)$ from these equations produces the sought for differential equation

$$(24.10)\qquad L(\psi) = \frac{(e^{-x}x^{\alpha}\psi)''}{e^{-x}x^{\alpha}} - \left(\alpha+1-x+\frac{2x\,Q''_{k+1}}{Q'_{k+1}}\right)\frac{(e^{-x}x^{\alpha}\psi)'}{e^{-x}x^{\alpha+1}} = -(n-k)\frac{\psi}{x}\,.$$

From (24.9) we readily conclude that ψ has a zero of order 2 at $x = 0$.

4. We now prove a converse proposition to (24.10) where Q_{k+1} is again Laguerre's polynomial. Specifically we show that the only entire

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10. Professor G. Latta has studied certain other differential equations satisfied by the determinants $\psi_n(x)$.

function $f(x)$ which possesses a zero of multiplicity at least 2 at $x=0$, satisfies

$$L(f) = \frac{\gamma f}{x}, \qquad x>0, \tag{24.11}$$

for some real $\gamma, \gamma \neq 0$, and it satisfies

$$\lim e^{-x} x^{h-k-1} f^{(h)}(x) = 0$$

for all $h=0, 1, 2, \ldots, x \to +\infty$, is necessarily a constant multiple of ψ_n for some n. In the case $\gamma = 0$ the growth condition must be

$$\lim e^{-x} x^{\alpha} f(x) = 0 \ , \ x \to +\infty,$$

and we conclude that $f(x) \equiv 0$.

For the proof, we note first that $f(x)$ being an entire function,

$$f(0) = f'(0) = 0,$$

the validity of (24.11) requires that $\dfrac{(e^{-x} x^{\alpha} f)'}{e^{-x} x^{\alpha+1}}$ contains all the zeros of Q'_{k+1}. Thus,

$$(e^{-x} x^{\alpha} f)' = e^{-x} x^{\alpha+1} Q'_{k+1} g \tag{24.12}$$

where g is again entire. We write (24.11) in the form

$$x^2 (Q'_{k+1})^2 \frac{d}{dx} \{u (e^{-x} x^{\alpha} f)'\} = \gamma f, \quad u = \frac{e^{x} x^{-\alpha-1}}{(Q'_{k+1})^2}, \tag{24.13}$$

$$x^2 (g' Q'_{k+1} - g Q''_{k+1}) = \gamma f,$$

so that inserting the last expression into (24.12)

$$\frac{d}{dx} \{e^{-x} x^{\alpha+2} (g' Q'_{k+1} - g Q''_{k+1})\} = \gamma e^{-x} x^{\alpha+1} Q'_{k+1} g$$

follows; or, equivalently,

$$x g'' + (\alpha + 2 - x) g' = (\gamma - k) g. \tag{24.14}$$

Now the argument is quite similar to that of § 23.4 (b). As to g we have to compare (23.18) and (24.14), i.e. α must be replaced by $\alpha + 1$. As to the growth condition to be imposed on $f(x)$ we compare (23.15) and (24.12) so that (apart from an immaterial change of α) Q_k must be replaced by $x^2 Q'_{k+1}$, i.e. k by $k+2$. Thus we conclude that g is a constant multiple of $Q_n^{(\alpha+1)} = \text{const.} (Q_{n+1}^{(\alpha)})' = \text{const.}\, Q'_{n+1}$ for some

appropriate integer n, $\gamma = k - n$. We find from (24.13), in view of (5.13), $\gamma \neq 0$, that $\frac{f}{x} = \text{const.}\ [(k+1)\,Q'_{n+1}\,Q_{k+1} - (n+1)\,Q_{n+1}\,Q'_{k+1}]$ so that [cf. (24.8)] $f = \text{const.}\ \psi_n$ as it was to be proved.

If $\gamma = 0$ we find from (24.13) that $g = \text{const.}\ Q'_{k+1}$ so that from (24.12)

$$e^{-x}\,x^{\alpha}\,f(x) = \text{const.} \int_0^x e^{-t}\,t^{\alpha+1}\,[Q'_{k+1}(t)]^2\,dt\,.$$

If we assume that $e^{-x}\,x^{\alpha}\,f(x) \to 0$ as $x \to +\infty$, $f \equiv 0$ follows.

5. (a) In the cases of the Hermite and Laguerre polynomials we have succeeded in characterizing the determinantal polynomials ψ_n as a complete set of solutions of an eigenvalue problem defined by a differential operator. It remains an open question whether it is possible to formulate an eigenvalue problem which determines ψ_n in the circumstance of the ultraspherical polynomials. At any rate we can obtain a second order differential equation satisfied by ψ_n which bears some similarity to the differential equation satisfied by the classical ultraspherical polynomial system.

Straightforward manipulations involving the identities (5.5), (5.6) lead to

$$\left(\frac{(\rho\psi)'}{\rho}\right)' + \frac{A}{\Delta}\,\frac{(\rho\psi)'}{\rho} = -\frac{(n-k)^2}{1-x^2}\,\psi + \frac{n-k}{\Delta}\,B\psi \tag{24.15}$$

for $\psi = \psi_n$ where

$$\rho = (1-x^2)^{\lambda-1/2}\ ,\quad \Delta = [Q_{k+1}]^2 + \frac{(1-x^2)\,[Q'_{k+1}]^2}{(k+1)^2}\,,$$

$$A = \frac{4\lambda\,Q_{k+1}\,Q'_{k+1}}{k+1} - \frac{4\lambda x\,[Q'_{k+1}]^2}{(k+1)^2}\,,$$

$$B = \frac{2\lambda\,[Q_{k+1}]^2}{1-x^2} - \frac{4\lambda x\,Q_{k+1}\,Q'_{k+1}}{(k+1)(1-x^2)} - \frac{2\lambda\,[Q'_{k+1}]^2}{(k+1)^2}\,.$$

This differential equation is of a different form than (24.10). Here the coefficients of ψ' and ψ'' are analytic on the interior of the interval of

orthogonality in contrast to (24.10) where the coefficient of ψ' has singularities (except when $k=0$) interior to this interval. Moreover the differential equation (24.15) does not generate the usual type of eigenvalue problem since it involves a term with the factor $(n-k)^2$ and a second term which is linear in $n-k$.

§ 25. Theorem 11, discrete measure, Wronski type, $l=2$.

The key property essential for the proof of Theorem 9 was the existence of a second order differential operator for which the underlying orthogonal polynomials are eigenfunctions. In this regard all systems of orthogonal polynomials satisfy a recursion formula which can be interpreted as a second order difference equation with respect to the integer variable n. Thus we may expect a result dual to Theorem 9 for the augmented determinantal systems of second order where the argument is the index variable n. This is Theorem 11 which we intend to discuss in the present section.

We assume that the spectrum is a countable discrete set

$$a_0 < a_1 < a_2 < \dots .$$

(Actually we could deal with the case of a finite discrete spectrum as well at the expense of a proliferation of cases.) For definiteness, we set $a_0 = 0$. The polynomials $\{Q_n(x)\}$ are normalized by the condition $Q_n(0)=1$ so that they satisfy a recursion of the form

$$-xQ_n(x) = -(\lambda_n+\mu_n)\,Q_n(x) + \lambda_n\,Q_{n+1}(x) + \mu_n\,Q_{n-1}(x), \qquad n = 0, 1, 2, \dots \tag{25.1}$$

where

$$Q_{-1}(x) \equiv 0\ ,\ Q_0(x) \equiv 1\ ,\ \lambda_n > 0\ ,\ \mu_n > 0$$

except $\mu_0 = 0$. We define

$$\pi_0 = 1\,,\quad \pi_n = \frac{\lambda_0\,\lambda_1 \dots \lambda_{n-1}}{\mu_1\,\mu_2 \dots \mu_n}\,,\qquad n \geqq 1\,, \tag{25.2}$$

so that

$$\mu_n\,\pi_n = \lambda_{n-1}\,\pi_{n-1}\ ;\ n \geqq 1\,.$$

Now we write (25.1) in the form

$$(25.3) \qquad -x\pi_n Q_n(x) = \lambda_n \pi_n \Delta Q_n(x) - \lambda_{n-1}\pi_{n-1}\Delta Q_{n-1}(x) = \Delta[\lambda_{n-1}\pi_{n-1}\Delta Q_{n-1}(x)], \qquad n \geqq 1,$$

where the difference operator in each case refers to the index n; i. e.

$$\Delta f_n = f_{n+1} - f_n .$$

Equation (25.3) expresses a Sturm-Liouville difference eigenvalue type relation in which x is an eigenvalue and $Q_n(x)$ represents the corresponding eigenfunction of argument n.

We shall deal with the augmented determinantal system

$$(25.4) \quad \Phi_r(k\,;\,n) = \Phi_r(n) = \begin{vmatrix} Q_n(a_k) & Q_n(a_r) \\ Q_{n+1}(a_k) & Q_{n+1}(a_r) \end{vmatrix} = \begin{vmatrix} Q_n(a_k) & Q_n(a_r) \\ \Delta Q_n(a_k) & \Delta Q_n(a_r) \end{vmatrix}, \quad r = k+1\,,\,k+2\,,\ldots;$$

concerning the notation used cf. the remark to Theorem 11 in the Introduction. The dependence on k will be suppressed since it is being held fixed. As usual, we consider for $y \geqq 0$ the linear extension

$$(25.5) \qquad \Phi_r(y) = \rho\Phi_r(n) + (1-\rho)\,\Phi_r(n+1),$$

where

$$y = \rho n + (1-\rho)(n+1)\ ,\ 0 \leqq \rho \leqq 1, \qquad n = 0, 1, 2, \ldots .$$

2. The proof of Theorem 11 is divided in a sequence of steps.

(a) $\Phi_{k+1}(n)$ is negative for all $n \geqq 0$. This fact is a particular consequence of Theorem 3.

(b) $\Phi_r(0)$ is negative for all $r = k+1\,,\,k+2,\ldots$. This is trivial.

(c) Two successive functions $\Phi_r(y)$ and $\Phi_{r+1}(y)$ can not vanish simultaneously.

In order to prove (c) we suppose $\Phi_r(y_0) = 0$; then there exists an index n and a value $\rho\,,\,0 \leqq \rho \leqq 1$, for which

$$(25.6) \qquad \Phi_r(y_0) = \begin{vmatrix} \rho Q_n(a_k) - (1-\rho)\,Q_{n+2}(a_k) & Q_{n+1}(a_k) \\ \rho Q_n(a_r) - (1-\rho)\,Q_{n+2}(a_r) & Q_{n+1}(a_r) \end{vmatrix} = 0,$$

$$y_0 = \rho n + (1-\rho)(n+1).$$

Thus

$$A\,Q_{n+1}(a_r) - B\,[\rho\,Q_n(a_r) - (1-\rho)\,Q_{n+2}(a_r)] = 0 \tag{25.7}$$

where

$$A = \rho\,Q_n(a_k) - (1-\rho)\,Q_{n+2}(a_k) \quad \text{and} \quad B = Q_{n+1}(a_k).$$

If $B = 0$ then $A \neq 0$ since in view of (25.1) we have

$$Q_n(a_k)\,Q_{n+2}(a_k) < 0.$$

Thus A and B can not vanish simultaneously. Now, if also $\Phi_{r+1}(y_0) = 0$ then

$$A\,Q_{n+1}(a_{r+1}) - B\,[\rho\,Q_n(a_{r+1}) - (1-\rho)\,Q_{n+2}(a_{r+1})] = 0. \tag{25.8}$$

The system (25.7), (25.8) implies

$$\rho Q\begin{pmatrix} n, & n+1 \\ a_r, & a_{r+1} \end{pmatrix} + (1-\rho)\,Q\begin{pmatrix} n+1, & n+2 \\ a_r, & a_{r+1} \end{pmatrix} = 0$$

in contradiction to Theorem 3; this establishes the assertion of (c).

(d) $\Phi_r(y)$ vanishes for at most two consecutive integer values of y, say n, $n+1$, and this happens if and only if

$$Q_{n+1}(a_k) = Q_{n+1}(a_r) = 0.$$

Let $\Phi_r(n) = \Phi_r(n+1) = 0$; then

$$\begin{vmatrix} \lambda\,Q_{n+2}(a_k) + \mu\,Q_n(a_k) & Q_{n+1}(a_k) \\ \lambda\,Q_{n+2}(a_r) + \mu\,Q_n(a_r) & Q_{n+1}(a_r) \end{vmatrix} = 0$$

for arbitrary λ and μ. Choosing λ and μ suitably with reference to the recursion (25.1) we find

$$\begin{vmatrix} a_k\,Q_{n+1}(a_k) & Q_{n+1}(a_k) \\ a_r\,Q_{n+1}(a_r) & Q_{n+1}(a_r) \end{vmatrix} = 0$$

and therefore either $Q_{n+1}(a_k) = 0$ or $Q_{n+1}(a_r) = 0$ or possibly both relations hold. Actually either relation implies the other in view of the hypothesis $\Phi_r(n) = 0$ since Q_n and Q_{n+1} can not have the common zero a_k. Finally $\Phi_r(n+2)$ can not also vanish since then by the previous argument $Q_{n+2}(a_k) = 0$ would follow.

Scrutiny of this reasoning reveals that a zero

$$y_0 = \rho n + (1-\rho)(n+1)$$

of $\Phi_r(y)$ lies in an interval of zeros of this function if and only if either $Q_{n+1}(a_k)=0$ or $Q_{n+1}(a_r)=0$ and consequently both necessarily hold.

3. We consider hereafter two cases.

(i) Let $y_0=\rho n+(1-\rho)(n+1)$ be an isolated zero of $\Phi_r(y)$. By virtue of (25.6) and the remark in (d) we have

$$Q_{n+1}(a_k)\neq 0\ ,\ Q_{n+1}(a^r)\neq 0$$

and

$$\rho\, Q_n(a_k)-(1-\rho)\, Q_{n+2}(a_k)=-\frac{Q_{n+1}(a_k)}{Q_{n+1}(a_r)}\,[\rho\, Q_n(a_r)-(1-\rho)\, Q_{n+2}(a_r)]. \tag{25.9}$$

We now simplify $\Phi_{r-1}(y_0)$ with the aid of (25.9) thus obtaining, cf. (25.6),

$$\Phi_{r-1}(y_0)=-\frac{Q_{n+1}(a_k)}{Q_{n+1}(a_r)}\left[\rho Q\begin{pmatrix} n, & n+1\\ a_{r-1}, & a_r\end{pmatrix}+(1-\rho)\, Q\begin{pmatrix} n+1, & n+2\\ a_{r-1} & a_r\end{pmatrix}\right]. \tag{25.10}$$

We prove in a quite similar fashion

$$\Phi_{r+1}(y_0)=\frac{Q_{n+1}(a_k)}{Q_{n+1}(a_r)}\left[\rho Q\begin{pmatrix} n, & n+1\\ a_r, & a_{r+1}\end{pmatrix}+(1-\rho)\, Q\begin{pmatrix} n+1, & n+2\\ a_r, & a_{r+1}\end{pmatrix}\right]. \tag{25.11}$$

The expressions in the square brackets are strictly negative (Theorem 3). Thus

$$\Phi_{r-1}(y_0)\cdot\Phi_{r+1}(y_0)<0\,. \tag{25.12}$$

Next, we evaluate the sign of

$$\Delta\,\Phi_r(n)=\Phi_r(n+1)-\Phi_r(n)\,.$$

To this end, we observe that

$$\Delta\,(p_n q_n)=p_{n+1}\,\Delta\, q_n+q_n\,\Delta\, p_n$$

so that for any function $r(n)$:

$$\Delta\,[r(n)\,\Phi_r(n)]=[(1-\rho)\, r(n)+\rho r(n+1)]\ \Delta\,\Phi_r(n)$$
$$+\,[\rho\,\Phi_r(n)+(1-\rho)\,\Phi_r(n+1)]\,\Delta\, r(n)$$

with arbitrary ρ. In particular we select ρ such that

$$\rho n+(1-\rho)\,(n+1)=y_0\,,\qquad 0\leq\rho\leq 1\,,$$

and

$$r(n) = \lambda_n \pi_n > 0$$

for all $n \geqq 0$. Since $\Phi_r(y_0) = 0$ it follows immediately that

$$\operatorname{sgn} \Delta [\lambda_n \pi_n \Phi_r(n)] = \operatorname{sgn} \Delta \Phi_r(n) . \tag{25.13}$$

We perform the calculation of the left hand side by using the second expression in (25.4) and taking (25.3) into account:

$$\Delta [\lambda_n \pi_n \Phi_r(n)] = \begin{vmatrix} Q_{n+1}(a_k) & Q_{n+1}(a_r) \\ \Delta [\lambda_n \pi_n \Delta Q_n(a_k)] & \Delta [\lambda_n \pi_n \Delta Q_n(a_r)] \end{vmatrix} \tag{25.14}$$

$$= - \pi_{n+1}(a_r - a_k) Q_{n+1}(a_k) Q_{n+1}(a_r) .$$

Relations (25.10), (25.11), (25.13) and (25.14) imply

$$\Phi_{r-1}(y_0) \Delta \Phi_r(n) < 0 \ , \ \Phi_{r+1}(y_0) \Delta \Phi_r(n) > 0 . \tag{25.15}$$

4. (ii) Let

$$y_0 = \rho n + (1-\rho)(n+1)$$

be a non-isolated zero of $\Phi_r(y)$. For definiteness we take y_0 in $[n, n+1]$, $\Phi_r(n) = \Phi_r(n+1) = 0$ and without loss of generality we identify $y_0 = n$. Then according to (d) we have necessarily

$$Q_{n+1}(a_k) = Q_{n+1}(a_r) = 0 .$$

In place of (25.10) and (25.11) we obtain

$$\Phi_{r-1}(n) = Q_n(a_k) Q_{n+1}(a_{r-1}) \ , \quad \Phi_{r+1}(n) = Q_n(a_k) Q_{n+1}(a_{r+1})$$

which we will convert now into a more useful form. Theorem 3 and $Q_{n+1}(a_r) = 0$ yield

$$Q_{n+1}(a_{r-1}) Q_n(a_r) > 0 , \quad Q_n(a_r) Q_{n+1}(a_{r+1}) < 0 .$$

Thus,

$$\operatorname{sgn} \Phi_{r-1}(n) = - \operatorname{sgn} \Phi_{r+1}(n) = \operatorname{sgn} [Q_n(a_k) Q_n(a_r)] . \tag{25.16}$$

With the aid of (25.1) it follows easily that

$$\operatorname{sgn} \Delta^2 \Phi_r(n) = \Phi_r(n+2) = - \operatorname{sgn} Q_{n+2}(a_k) Q_{n+2}(a_r) .$$

Thus, the sign of $\Phi_{r-1}(n) \Delta^2 \Phi_r(n)$ is that of

$$- [Q_n(a_k) Q_{n+2}(a_k)] [Q_n(a_r) Q_{n+2}(a_r)]$$

which is negative since each of the bracketed factors is negative. To sum up: in the case of a non-isolated zero $y_0 = n$, we obtain the relations

$$(25.17)\qquad \begin{aligned} &\Phi_{r-1}(n)\,\Delta\,\Phi_r(n) = 0 \;,\; \Phi_{r-1}(n)\,\Delta^2\,\Phi_r(n) < 0\,, \\ &\Phi_{r+1}(n)\,\Delta\,\Phi_r(n) = 0 \;,\; \Phi_{r+1}(n)\,\Delta^2\,\Phi_r(n) > 0\,. \end{aligned}$$

5. In accordance with the procedures of § 2 we need to determine the sign of $\Phi_r(n)$ for n large. To this end, we make use of the following results to be established in Appendix, § 31 (cf. Theorem 19 and its Corollary).

First of all, when n is sufficiently large $Q_n(a_r)$ has the constant sign $(-1)^r$. Since always $Q\begin{pmatrix} n\;,\; n+1 \\ a_s\;,\; a_{s+1}\end{pmatrix} < 0$ (Theorem 3) we infer moreover for all $n > N(a_s\,,\, a_{s+1})$, $N(a_s\,,\,a_{s+1})$ is an appropriate constant depending on a_s and a_{s+1}, the inequality

$$(25.18)\qquad \frac{Q_n(a_s)}{Q_{n+1}(a_s)} - \frac{Q_n(a_{s+1})}{Q_{n+1}(a_{s+1})} > 0\,.$$

Now fix a_k and a_r, $a_k < a_r$; then for n sufficiently large, specifically for

$$n > \max\,\{N(a_k\,,\,a_{k+1})\;,\; N(a_{k+1}\,,\,a_{k+2})\,,\,\ldots\,,\,N(a_{r-1}\,,\,a_r)\}\,,$$

we obtain by adding the inequalities (25.18) for $s = k\,,\,k+1,\ldots, r-1$ that

$$(25.19)\qquad \frac{Q_n(a_k)}{Q_{n+1}(a_k)} > \frac{Q_n(a_r)}{Q_{n+1}(a_r)}\,.$$

On the basis of the facts mentioned we conclude that $\Phi_r(k\,;\,n) = \Phi_r(n)$ has the constant sign $(-1)^{k+r}$.

With the aid of the statements (a)—(d), of the basic inequalities (25.15) and (25.17), and by virtue of the previous remarks we deduce in a routine manner Theorem 11.

§ 26. Theorem 12, classical polynomials, higher (even) order Wronskians.

1. In generalizing Theorem 9 we shall deal now with the Sturmian properties of the sets of higher order augmented Wronskians having the special form

$$(26.1)\quad \varphi_n(k\,,l\,;x) = \varphi_n(x) = W(Q_k(x)\,,Q_n(x)\,,Q_{n+1}(x)\,,\ldots,Q_{n+l}(x))\,,\qquad n = k+1\,,k+2\,,\ldots.$$

The corresponding matrix is of size $(l+2)\times(l+2)$ where l is a fixed even integer; k is an arbitrary fixed integer and $\{Q_n(x)\}$ is one of the classical orthogonal polynomials systems defined in Theorem 5. (Henceforth, the dependence of φ_n on k and l will be suppressed without causing any ambiguities.) We shall rely decisively on the familiar fact that the system $\{Q_n(x)\}$ is characterized as a set of eigenfunctions for a differential equation of the form

$$(26.2)\qquad r(x)\,Q''_n(x) + s(x)\,Q'_n(x) = \lambda_n\,Q_n(x)\,,\qquad n = 0,\,1,\,2,\ldots.$$

[The notation is slightly different from (23.1).] The boundary condition which determines $Q_n(x)$ uniquely is that the solution be an entire function of x. The eigenvalues λ_n are a strictly *decreasing* sequence of negative numbers; $r(x)$ and $s(x)$ are analytic in the interior of the interval of orthogonality (spectrum) and $r(x)>0$.

2. For the convenience of the exposition we divide the analysis of the set $\{\varphi_n(x)\}$ into several steps. We mention first the following two facts.

(a) *For real x exterior to the interval of orthogonality, including end points, the function $\varphi_n(x)$, $n\geq k+1$, never vanishes.*

(b) *The initial function $\varphi_{k+1}(x)$ has a single sign for all real x.*

(a) is a particular case of a more general theorem proved in the Appendix, § 32; (b) follows as an application of Theorem 1.

3. Let x_0 be a real zero of $\varphi_n(x)$. Then there exist certain real constants a, $\gamma_0\,,\gamma_1\,,\ldots,\gamma_l$ not all zero such that setting

$$(26.3)\qquad f(x) = a\,Q_k(x) + \sum_{\nu=0}^{l}\gamma_\nu\,Q_{n+\nu}(x)$$

we have the relations:

$$(26.4)\qquad f(x_0) = f'(x_0) = \ldots = f^{(l+1)}(x_0) = 0\,.$$

With the aid of the differential equation (26.2) we infer that

$$(26.5)\quad g(x) = r(x) f''(x) + s(x) f'(x) = a \lambda_k Q_k(x) + \sum_{\nu=0}^{l} \gamma_\nu \lambda_{n+\nu} Q_{n+\nu}(x).$$

Moreover $g^{(\mu)}(x)$ is a linear combination of the $f^{(\nu)}(x)$ with $\nu \leqq \mu + 2$ so that in view of (26.4)

$$(26.6)\qquad g(x_0) = g'(x_0) = \dots = g^{(l-1)}(x_0) = 0 .$$

In addition from (26.5)

$$(26.7)\qquad g^{(l)}(x_0) = r(x_0) f^{(l+2)}(x_0) .$$

We may write these relations in the following form:

$$(26.8)\qquad g^{(\mu)}(x_0) - \lambda_k f^{(\mu)}(x_0) = \sum_{\nu=0}^{l} \gamma_\nu (\lambda_{n+\nu} - \lambda_k) Q^{(\mu)}_{n+\nu}(x_0)$$
$$= \begin{cases} 0 & \text{for } \mu < l \\ r(x_0) f^{(l+2)}(x_0) & \text{for } \mu = l . \end{cases}$$

4. First we point out that $a \neq 0$. If, to the contrary, $a = 0$, then the constants $\gamma_0, \gamma_1, \dots, \gamma_l$ are not all zero and using (26.4) we see that the columns of the matrix

$$(Q^{(\mu)}_{n+\nu}(x_0)), \qquad \mu = 0, 1, \dots, l+1 \ ; \ \nu = 0, 1, \dots, l,$$

are linearly dependent. This implies that the Wronskian

$$(26.9)\qquad W = W(Q_n, Q_{n+1}, \dots, Q_{n+l})$$

and its derivative would vanish for $x = x_0$. The order $l + 1$ of this determinant being odd, this is in contradiction with the conclusions of Theorem 2. Thus $a \neq 0$; we may assume that $a = 1$.

We show that $\gamma_l \neq 0$. Assuming the contrary, we consider the system (26.8) but only with $\mu < l$: this system is homogeneous and its determinant, a Wronskian of even order, is not zero. Thus all γ_ν must be zero; but then (26.4) is a contradiction since $Q_k(x)$ has only simple zeros. We may prove similarly that $\gamma_0 \neq 0$.

5. We write the determinant (26.9) in the form $W = [w_{\mu\nu}]$ where μ, ν run from 0 to l; denoting its minors by $W_{\mu\nu}$ we may solve (26.8) in the following form:

$$W \cdot \gamma_\nu (\lambda_{n+\nu} - \lambda_k) = r(x_0) f^{(l+2)}(x_0) \cdot W_{l\nu}\,; \tag{26.10}$$

in particular for $\nu = 0$ and $\nu = l$

$$\begin{cases} W \cdot \gamma_0 (\lambda_n - \lambda_k) = r(x_0) f^{(l+2)}(x_0) \cdot W(Q_{n+1}, \ldots, Q_{n+l}), \\ W \cdot \gamma_l (\lambda_{n+l} - \lambda_k) = r(x_0) f^{(l+2)}(x_0) \cdot W(Q_n, \ldots, Q_{n+l-1}). \end{cases} \tag{26.11}$$

On the other hand, writing (26.8), $\mu < l$, as follows:

$$\sum_{\nu=0}^{l-1} \gamma_\nu (\lambda_{n+\nu} - \lambda_k) Q_{n+\nu}^{(\mu)}(x_0) = -\gamma_l (\lambda_{n+l} - \lambda_k) Q_{n+l}^{(\mu)}(x_0)$$

we may solve again and obtain, in particular,

$$\gamma_0 (\lambda_n - \lambda_k) \cdot W(Q_n, \ldots, Q_{n+l-1}) = \gamma_l (\lambda_{n+l} - \lambda_k) \cdot W(Q_{n+1}, \ldots, Q_{n+l}).$$

In all these equations $x = x_0$. As a special consequence (Theorem 3) we point out that $\gamma_0 \gamma_l > 0$.

6. Now we discuss

$$\varphi_{n-1}(x) = W(Q_k, Q_{n-1}, \ldots, Q_{n+l-1})\ ;\ \varphi_{n+1}(x) = W(Q_k, Q_{n+1}, \ldots, Q_{n+l+1})$$

for $x = x_0$ assuming again that $\varphi_n(x_0) = 0$. With the aid of (26.4) we combine the columns

$$Q_k, Q_n, Q_{n+1}, \ldots, Q_{n+l-1} \quad \text{and} \quad Q_k, Q_{n+1}, Q_{n+2}, \ldots, Q_{n+l}$$

and obtain

$$\varphi_{n-1}(x_0) = \gamma_l \cdot W(Q_{n-1}, Q_n, \ldots, Q_{n+l}) \tag{26.12}$$

as well as

$$\varphi_{n+1}(x_0) = -\gamma_0 \cdot W(Q_n, Q_{n+1}, \ldots, Q_{n+l+1}), \tag{26.13}$$

respectively. Both Wronskians are different from zero and of the same sign so that

$$\varphi_{n-1}(x_0)\, \varphi_{n+1}(x_0) < 0. \tag{26.14}$$

7. Finally we proceed to the more complicated discussion of $\varphi'_n(x_0)$; differentiation of the determinant $\varphi_n(x)$ raises merely the order of derivatives in the last row by one. Using the combination of the columns defined by (26.3), (26.4) we can make all elements in the first column to vanish except the last one which will become $f^{(l+2)}(x_0)$. Its minor being

$-W(Q_n, \ldots, Q_{n+l})$ we obtain, in view of the second equation (26.11)

$$\varphi'_n(x_0) = -f^{(l+2)}(x_0) \cdot W(Q_n, \ldots, Q_{n+l})$$
$$= -\frac{\gamma_l(\lambda_{n+l}-\lambda_k)}{r(x_0)} \cdot \frac{[W(Q_n, \ldots, Q_{n+l})]^2}{W(Q_n, \ldots, Q_{n+l-1})}.$$

Combining this with (26.12) we find

$$\varphi_{n-1}(x_0)\,\varphi'_n(x_0) = -\frac{\gamma_l^2(\lambda_{n+l}-\lambda_k)}{r(x_0)} \cdot \frac{W(Q_{n-1}, \ldots, Q_{n+l})}{W(Q_n, \ldots, Q_{n+l-1})} \cdot [W(Q_n, \ldots, Q_{n+l})]^2.$$

Here $\lambda_{n+l}-\lambda_k < 0$ and the two first Wronskians are of even order and opposite signs. Thus

(26.15) $\quad \varphi_{n-1}(x_0)\,\varphi'_n(x_0) < 0 \quad$ provided $\quad W = W(Q_n, Q_{n+1}, \ldots, Q_{n+l}) \neq 0$.

8. Now let

$$W = W(Q_n, \ldots, Q_{n+l}) = 0$$

for $x = x_0$ so that

$$\varphi_n(x_0) = \varphi'_n(x_0) = 0.$$

We prove then that

$$\varphi''_n(x_0) = 0 \;, \;\; \varphi'''_n(x_0) \neq 0,$$

and

(26.16) $$\varphi_{n-1}(x_0)\,\varphi'''_n(x_0) < 0.$$

Differentiating the determinant $\varphi_n(x)$ p-times modifies only the last p rows. We shall indicate these changes merely by the last pertinent elements of the first column involving the derivatives of Q_k; the corresponding elements of the later columns arise by replacing k by n, $n+1, \ldots, n+l$:

$$\varphi'_n = [Q_k^{(l+2)}]\,;\; \varphi''_n = [Q_k^{(l+1)}, Q_k^{(l+2)}] + [Q_k^{(l)}, Q_k^{(l+3)}],$$
$$\varphi'''_n = [Q_k^{(l)}, Q_k^{(l+1)}, Q_k^{(l+2)}] + 2\,[Q_k^{(l-1)}, Q_k^{(l+1)}, Q_k^{(l+3)}]$$
$$+ [Q_k^{(l-1)}, Q_k^{(l)}, Q_k^{(l+4)}].$$

Using the linear combination of the columns defined by (26.4) to which now $f^{(l+2)}(x_0) = 0$ has to be added (cf. (26.11), the determinants on the right are of even order hence not zero), we can reduce all elements in the

first column of the first determinant of φ_n'' to zero while in the second determinant of φ_n'' all but the last element will become zero. But the minor of that last element is $-W=0$ hence $\varphi_n''=0$.

The same procedure renders all elements in the first column of the three determinants making up φ_n''' zero except the last elements in the second and third determinant. Hence we obtain

$$(26.17) \qquad \varphi_n'''(x_0) = -2f^{(l+3)}(x_0)\cdot\frac{dW}{dx} - f^{(l+4)}(x_0)\cdot W$$
$$= -2f^{(l+3)}(x_0)\cdot\frac{dW}{dx}.$$

We note that $\dfrac{dW}{dx}\neq 0$ since $W=0$ (Theorem 2).

We modify now the system (26.8) by replacing the equation $\mu=l$ by $\mu=l+1$ in which case the right hand side must be $r(x_0)\,f^{(l+3)}(x_0)$. The determinant of this system will be $\dfrac{dW}{dx}$, thus (26.10), (26.11) hold without change except that W on the left and $f^{(l+2)}(x_0)$ on the right must be replaced by $\dfrac{dW}{dx}$ and $f^{(l+3)}(x_0)$, respectively. (The minors $W_{l\nu}$ do not change.) We find

$$(26.18) \qquad \frac{dW}{dx}\cdot\gamma_l(\lambda_{n+l}-\lambda_k) = r(x_0)\,f^{(l+3)}(x_0)\cdot W(Q_n,\ldots,Q_{n+l-1}).$$

From (26.17) and (26.18),

$$\varphi_n'''(x_0) = -\frac{2\gamma_l(\lambda_{n+l}-\lambda_k)}{r(x_0)}\;\frac{[dW/dx]^2}{W(Q_n,\ldots,Q_{n+l-1})},$$

$$\varphi_{n-1}(x_0)\,\varphi_n'''(x_0) = -\frac{2\gamma_l^2(\lambda_{n+l}-\lambda_k)}{r(x_0)}\cdot\frac{W(Q_{n-1},\ldots,Q_{n+l})}{W(Q_n,\ldots,Q_{n+l-1})}$$
$$\cdot\left\{\frac{d}{dx}W(Q_n(x),\,Q_{n+1}(x),\ldots,Q_{n+l}(x))\right\}^2.$$

We now have all the ingredients from which to verify the Sturm properties of the set $\{\varphi_n(x)\}$. This establishes the proof of Theorem 12.

§27. Theorem 13, discrete measure, higher (even) order Wronskians.

1. A discrete analog of the determinants (26.1) of §26 are the quantities

$$(27.1) \quad \psi_r(k\,,\,l\,;\,n) = \psi_r(n) = Q\begin{pmatrix} a_k\,, a_r\,, & a_{r+1}\,, \ldots, a_{r+l} \\ n\,, n+1\,, & n+2\,, \ldots, n+l+1 \end{pmatrix}; \qquad r = k+1\,, k+2\,, \ldots,$$

where we follow the notation (1.3′). The corresponding matrix is again of size $(l+2)\times(l+2)$, l a fixed even integer, k a fixed integer, and $\{Q_n(x)\}$ constitute an arbitrary system of orthogonal polynomials associated with a discrete measure whose spectrum is the discrete set $a_0 < a_1 < a_2 \ldots$. Our aim is to prove Theorem 13, i.e. that the sequences $\{\psi_r(n)\,;\,n = 0, 1, 2, \ldots\}$ form a weak Sturm set in r; $r = k+1,\ k+2, \ldots$ (see §2). With other words, the sequence mentioned will have $r-k-1$ nodal intervals, and the nodal intervals of the sequences r and $r+1$ interlace. The nodal intervals will not contain more than two integer points.

The special case $l = 2$ has been discussed in Theorem 11 proved in §25. By using the method of §10 we are in the position to reduce the general situation (27.1) to that of §25, i.e. to the determinants denoted there by $\Phi_r(n)$. We assume again that

$$a_0 = 0\ ,\ Q_n(0) = 1\,.$$

2. Let r be fixed. We consider the new system of polynomials

$$q_n(r\,;\,x) = q_n(x)$$

defined by the formula

$$(27.2) \quad Q\begin{pmatrix} x\,, a_{r+1}\,, a_{r+2}\,, \ldots, a_{r+l} \\ n\,, n+1\,, n+2, \ldots, n+l \end{pmatrix} = (-1)^{l/2}(x-a_{r+1})(x-a_{r+2})\ldots(x-a_{r+l})\cdot q_n(x)\,.$$

This expression is a linear combination of the polynomials $Q_n(x), \ldots, Q_{n+l}(x)$ having the form indicated on the right and $q_n(x)$ is of the precise degree n since the coefficient of $Q_{n+l}(x)$ has the sign $(-1)^{l/2}$ (Theorem 3). Also $q_n(0) > 0$. The polynomials $\{q_n(x)\}$ are orthogonal on a spectrum which is identical with the previous one except that the points $a_{r+1}, \ldots, a_{r+l}$ are

missing; hence a_r and a_{r+l+1} are successive points of the new spectrum. Applying Theorem 11 to the new system $\{q_n(x)\}$ we obtain that the sequence

$$(27.3) \qquad \begin{vmatrix} q_n(a_k) & q_n(a_r) \\ q_{n+1}(a_k) & q_{n+1}(a_r) \end{vmatrix}; \qquad n = 0, 1, 2, \dots,$$

has $r-k-1$ nodal intervals; moreover the sequence

$$(27.4) \qquad \begin{vmatrix} q_n(a_k) & q_n(a_{r+l+1}) \\ q_{n+1}(a_k) & q_{n+1}(a_{r+l+1}) \end{vmatrix}; \qquad n = 0, 1, 2, \dots,$$

has $r-k$ nodal intervals. The nodal intervals of the sequences (27.3) and (27.4) interlace.

3. On the other hand, applying Sylvester's theorem to the determinant (27.1) (we strike out the first and last row and the two first columns) we obtain

$$(27.5) \qquad \psi_r(n) \cdot Q\begin{pmatrix} a_{r+1}, \dots, a_{r+l} \\ n+1, \dots, n+l \end{pmatrix} = c \begin{vmatrix} q_{n+1}(a_r) & q_{n+1}(a_k) \\ q_n(a_r) & q_n(a_k) \end{vmatrix},$$

$$c = (a_r - a_{r+1}) \dots (a_r - a_{r+l}) \cdot (a_k - a_{r+1}) \dots (a_k - a_{r+l}) > 0 .$$

Applying further the same theorem to the determinant $\psi_{r+1}(n)$ (we strike out now the first and last row and the first and last column) we find

$$(27.6) \qquad \psi_{r+1}(n) \cdot Q\begin{pmatrix} a_{r+1}, \dots, a_{r+l} \\ n+1, \dots, n+l \end{pmatrix} = c' \begin{vmatrix} q_{n+1}(a_{r+l+1}) & q_{n+1}(a_k) \\ q_n(a_{r+l+1}) & q_n(a_k) \end{vmatrix},$$

$$c' = (a_{r+l+1} - a_{r+1}) \dots (a_{r+l+1} - a_{r+l}) \cdot (a_k - a_{r+1}) \dots (a_k - a_{r+l}) > 0 .$$

After having removed the first column of $\psi_{r+1}(n)$ we have to transfer the last column to the first place which is equivalent with multiplication by $(-1)^l = 1$. In (27.5) and (27.6) the quantities Q on the left are of the sign $(-1)^{l/2}$.

In view of what we know about (27.3) and (27.4) the assertion follows with the aid of (27.5) and (27.6) immediately.

§ 28. Theorem 14, classical polynomials, a special Turán system.

As announced in the Introduction we shall evaluate now in explicit terms the determinant T of the Turán type defined in Theorem 14; this will be done in all cases of classical polynomials $Q_n(x)$ occuring in

Theorem 5. The case of Legendre's polynomials with $r = l-1$ has been treated already in § 4.2 as an illustration of a simple method to be used now also. The case of the Legendre and ultraspherical polynomials with $r = l-1$ is taken care of by the identities (4.9), (12.3), (14.3).

1. Legendre polynomials, $r \geqq l-1$. We follow the method as well as the notation of § 4.2. We evaluate the determinant

$$[A] = T(P_0(x), P_1(x), \ldots, P_{l-2}(x), P_r(x)) \tag{28.1}$$

where $r \geqq l-1$. For $r = l-1$ this reduces to (4.3). For $r \geqq l$ the determinant in question is non-symmetric.

Assuming $r \geqq l$, we choose $H = (h_{pq})$ as in § 4.2. Let first $0 \leqq q \leqq l-2$ so that $\nu \leqq q \leqq l-2$; hence $a_{\mu\nu} = P_{\mu+\nu}$ for all μ. This leads to the same quantities b_{pq}, $q \leqq l-2$, as in § 4, see (4.8). Thus all elements of B under the main diagonal vanish and it suffices to compute b_{pq} for $p = q = l-1$. We have

$$b_{l-1,l-1} = \sum h_{\mu,l-1} a_{\mu\nu} h_{\nu,l-1} = \sum_{\mu \leqq l-1,\, \nu \leqq l-2} + \sum_{\mu \leqq l-1,\, \nu = l-1} = \sum_1 + \sum_2 . \tag{28.2}$$

In the first sum Σ_1 we have $a_{\mu\nu} = P_{\mu+\nu}$, in the second sum Σ_2 we have $a_{\mu\nu} = P_{\mu+r}$. Consequently

$$\sum_1 = \frac{1}{\pi} \int_0^\pi \sum_{\mu=0}^{l-1} h_{\mu,l-1} (x + \sqrt{x^2-1} \cos\varphi)^\mu \cdot \sum_{\nu=0}^{l-2} h_{\nu,l-1} (x + \sqrt{x^2-1} \cos\varphi)^\nu \cdot d\varphi$$

$$= \frac{1}{\pi} \int_0^\pi \cos(l-1)\varphi \cdot \sum_{\nu=0}^{l-2} h_{\nu,l-1} (x + \sqrt{x^2-1} \cos\varphi)^\nu \cdot d\varphi = 0,$$

$$\sum_2 = \sum_{\mu=0}^{l-1} h_{\mu,l-1} P_{\mu+r} h_{l-1,l-1}$$

$$= h_{l-1,l-1} \cdot \frac{1}{\pi} \int_0^\pi \sum_{\mu=0}^{l-1} h_{\mu,l-1} (x + \sqrt{x^2-1} \cos\varphi)^{\mu+r} d\varphi$$

$$= h_{l-1,l-1} \cdot \frac{1}{\pi} \int_0^\pi \cos(l-1)\varphi \cdot (x + \sqrt{x^2-1} \cos\varphi)^r d\varphi$$

$$= h_{l-1,l-1} \cdot \frac{r!}{(r+l-1)!} \cdot (x^2-1)^{\frac{l-1}{2}} \left(\frac{d}{dx}\right)^{l-1} P_r(x)$$

$$= 2^{l-2} \cdot (x^2-1)^{-\frac{l-1}{2}} \cdot \frac{r!}{(r+l-1)!} (x^2-1)^{\frac{l-1}{2}} \left(\frac{d}{dx}\right)^{l-1} P_r(x) .$$

Here we used (4.7) and (5.9). Now $[A]$ can be evaluated as in (4.9); all quantities h_{pp}, b_{tp} are the same as there, only $b_{l-1,l-1}$ has to be altered. Making use of the result (4.9), we conclude in the present case

$$\begin{aligned} [A] &= 2^{-(l-1)^2} (x^2-1)^{l(l-1)/2} \cdot 2b_{l-1,l-1} \\ &= 2^{-(l-1)(l-2)} (x^2-1)^{l(l-1)/2} \cdot \frac{r!}{(r+l-1)!} \left(\frac{d}{dx}\right)^{l-1} P_r(x) . \end{aligned} \tag{28.3}$$

Thus the polynomial $[A]$ changes its sign exactly $r-l+1$ times in the interval $-1 < x < +1$.

For $r = l$ this formula was deduced by [Beckenbach-Seidel-Szász 3]:

$$[A] = 2^{-(l-1)^2} l (x^2-1)^{l(l-1)/2} x . \tag{28.4}$$

2. Ultraspherical polynomials. Now we evaluate

$$[A] = T(Q_0(x), Q_1(x), \ldots, Q_{l-2}(x), Q_r(x)), \quad Q_n(x) = Q_n = \frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)}, \tag{28.5}$$

$r \geqq l-1$, using the integral representation (5.8). As to the case $r = l-1$ cf. (14.3). Dealing with the general case $r \geqq l$ we follow a similar argument as in **1**; the notation is similar to that in § 4.2. For the sake of simplicity we assume that $\lambda > \frac{1}{2}$ and $x > 1$. In the present case we use, instead of (4.6), the polynomials

$$h_p(t) = \sum_{\mu=0}^{p} h_{\mu p} (x + \sqrt{x^2-1}\, t)^{\mu} = \left(\pi^{-\frac{1}{2}} \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)} h_p^{(\lambda-\frac{1}{2})}\right)^{-\frac{1}{2}} P_p^{(\lambda-\frac{1}{2})}(t) \tag{28.6}$$

normalized by the condition (14.6), $n = 0$, see (5.7). The details are clearly the same as in **1**; we have to compute only b_{pq}, $p = q = l-1$ for which we write as before $\Sigma_1 + \Sigma_2$.

Again $\Sigma_1 = 0$ and in view of (5.8),

$$\sum_2 = \sum_{\mu=0}^{l-1} h_{\mu,l-1}\, Q_{\mu+r}\, h_{l-1,l-1} = h_{l-1,l-1} \cdot \pi^{-\frac{1}{2}} \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)}$$

$$\cdot \int_0^\pi h_{l-1}(\cos\varphi)\,(x+\sqrt{x^2-1}\cos\varphi)^r \sin^{2\lambda-1}\varphi\, d\varphi$$

$$= (x^2-1)^{-\frac{l-1}{2}} \cdot \frac{k_{l-1}^{(\lambda-\frac{1}{2})}}{h_{l-1}^{(\lambda-\frac{1}{2})}} \int_0^\pi P_{l-1}^{(\lambda-\frac{1}{2})}(\cos\varphi)\,(x+\sqrt{x^2-1}\cos\varphi)^r \sin^{2\lambda-1}\varphi\, d\varphi\,.$$

Here we compared the leading terms in (28.6); see (5.4). Using (5.10) we see that the latter expression is, except for constant factors, identical with $\left(\frac{d}{dx}\right)^{l-1} P_r^{(\lambda)}(x)$.

We refer again to (14.3) (which is the case $r = l-1$) and notice that compared with this case only $b_{l-1,l-1}$ has to be changed. (With other words, the quotient $[A]/b_{l-1,l-1}$ is independent of r.) Thus

$$[A] = C\,(x^2-1)^{l(l-1)/2} \left(\frac{d}{dx}\right)^{l-1} P_r^{(\lambda)}(x)$$

where the constant C is different from zero and depends on the parameters λ, l, r.

3. Laguerre polynomials. We deal further with

(28.7) $[A] = T\,(Q_0(x), Q_1(x), \ldots, Q_{l-2}(x), Q_r(x)),\ Q_n(x) = Q_n^{(\alpha)}(x) = \dfrac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}$,

$r \geqq l-1$. As to the case $r = l-1$, cf. (16.5). In the present case we assume $r \geqq l$ and follow a similar notation and a similar argument as in **2**. The polynomials (16.14) will be instrumental, $n = 0$, so that we can choose

$$h_p(z) = \text{const.}\ f_p(z)\,, \qquad h_{pp} = \text{const.}\ x^{-p}$$

where the latter constant is independent of x; cf. (16.17) and (16.18). We have as in (16.15):

$$b_{l-1,l-1} = \sum_{\mu=0}^{l-1} h_{\mu,l-1}\, Q_{\mu+r}^{\alpha}(x)\, h_{l-1,l-1} = \sum_{\rho=0}^{r} k_\rho(x) \sum_{\mu=0}^{l-1} h_{\mu,l-1} \binom{\mu+r}{\rho} h_{l-1,l-1}$$
$$= h_{l-1,l-1} \cdot K\{(1+z)^r h_{l-1}(z)\} = \text{const.}\ h_{l-1,l-1} \cdot K\{(1+z)^r f_{l-1}(z)\}.$$

Now we use (16.22) and obtain, in view of (16.5) (see also the first part of (5.14))

$$(28.8)\quad [A] = C\, x^{l(l-1)}\, Q_{r-l+1}^{(\alpha+2l-2)}(x) = C'\, x^{l(l-1)} \left(\frac{d}{dx}\right)^{l-1} Q_r^{(\alpha+l-1)}(x)$$

where the constants C and C' are different from zero and depend on α, l, r.

4. Hermite polynomials. This case is trivial since in view of (18.3):

$$[A] = T(H_0(x), H_1(x), \dots, H_{l-2}(x), H_r(x)) = (-1)^{l(l-1)/2} \cdot W(H_0(x), H_1(x), \dots, H_{l-2}(x), H_r(x))$$

so that

$$(28.9)\qquad [A] = C\left(\frac{d}{dx}\right)^{l-1} H_r(x), \qquad C \neq 0.$$

§29. Theorem 15, discrete measure, a special Wronski system.

The determinant of the discrete type $\psi(r, l; n) = \psi_r(n)$ which occurs in Theorem 15 can be described as follows. In the first row we have the vector

$$(29.1)\qquad \{Q_n(a_0), Q_n(a_1), \dots, Q_n(a_{l-2}), Q_n(a_r)\}, \qquad r \geqq l-1,$$

and the later rows arise by replacing n by $n+1, n+2, \dots, n+l-1$, respectively. In our general interpretation it is dual to the Wronskian

$$(29.2)\quad \varphi(r, l; x) = \varphi_r(x) = W(Q_0(x), Q_1(x), \dots, Q_{l-2}(x), Q_r(x)).$$

The latter is of course trivially equal to const. $\left(\frac{d}{dx}\right)^{l-1} Q_r(x)$, and this is evidently true for any system of orthogonal polynomials (even more generally). The simplification in (29.2) is due to the triangular nature of the determinant.

1. We make the following simple observations.

If $\{Q_r(x)\}$ is one of the familiar systems of classical orthogonal polynomials, $\left(\frac{d}{dx}\right)^{l-1} Q_r(x)$ will be again a system of orthogonal poly-

nomials. In particular, $\varphi_r(x)$, $r \geqq l-1$, comprise a Sturm set on an appropriate interval. Probably the classical polynomials are the only systems of this property. We refer to a known result [13, p. 106] which asserts that $\{Q_r(x)\}$ and $\{Q'_r(x)\}$ are both orthogonal systems of polynomials if and only if $\{Q_n(x)\}$ is one of the classical types.

It is relatively easy to construct examples of general orthogonal polynomials for which $\{Q'_r(x)\}$, $r = 1, 2, \ldots$, is not a Sturm set. Thus we see that the Sturm property of an orthogonal system of polynomials is not automatically transmitted to the derived system. Some additional hypothesis is needed. It may by conjectured that only the classical polynomials possess the property that $\varphi_r(x)$ as defined by (29.2) is a Sturm set for every l.

On the other hand we emphasized in § 25 that the polynomials $\{Q_n(x)\}$, for each x regarded as functions in the variable n satisfy a Sturm-Liouville type of difference equation represented by the recursion law [cf. (25.1)]. In this sense all orthogonal polynomials behave like the classical systems. This suggests the possibility of establishing the Sturm properties of $\varphi_r(x)$ with respect to the variable r in general as we generate successive functions by different choices of x. At present we consider the dual quantity $\psi_r(n)$ and we concentrate our attention to a system of orthogonal polynomials with respect to a measure with a discrete spectrum.

2. We denote the spectrum by $a_0 < a_1 < a_2 < \ldots$. We want to prove the Sturm characteristics of the sequences $\psi_r(n)$: $n = 0, 1, 2, \ldots$, defined by (1.26) or (29.1); here $r \geqq l-1$. We note that $\psi_r(n)$ can be written in an alternative form where in the second row we replace Q_{n+1} by ΔQ_n, in the third row Q_{n+2} by $\Delta^2 Q_n$, etc.; the difference operations are always executed with respect to the variable n. Pertaining to the properties of the zeros of these functions, the normalization of Q_n is clearly irrelevant.

The essential tool is a reduction which is analogous to that of (29.2). It follows a similar idea as the one used in proving the formula of Christoffel [13, p. 29].

Let $\{Q_n(x)\}$ be an orthogonal system of polynomials with respect to the distribution $d\alpha(x)$; the system $\{Q_n^{[l]}(x)\}$ formed according to the rule

$$Q_n^{[1]}(x) = \frac{\dfrac{Q_n(x)}{Q_n(a_0)} - \dfrac{Q_{n+1}(x)}{Q_{n+1}(a_0)}}{x - a_0} \tag{29.3}$$

will be orthogonal with respect to the distribution $(x-a_0)\,d\,\alpha(x)$. We call $\{Q_n^{[1]}(x)\}$ the first associate system of $\{Q_n(x)\}$.

The proof is straightforward. In particular, if the spectral set of the given system is $a_0 < a_1 < a_2 < \dots$, the associate system will have the spectral set $a_1 < a_2 < a_3 \dots$. In other words the first point of the spectrum ceases to be a spectral point in passing from Q_n to the first associate system $Q_n^{[1]}$. Continuing in this manner we may define the k^{th} associate system of Q_n denoted by $Q_n^{[k]}$. The spectrum of $Q_n^{[k]}$ will be the set $a_k < a_{k+1} < a_{k+2} < \dots$.

3. We are now prepared to convert $\psi_r(n)$ defined by (29.1) into a more transparent form. We divide the single rows of $\psi_r(n)$ by $Q_n(a_0)$, $Q_{n+1}(a_0), \dots, Q_{n+l-1}(a_0)$, respectively. Subtracting the second row from the first, the third row from the second etc. and dividing by

$$a_1 - a_0\,,\ a_2 - a_0\,,\ \dots,\ a_{l-2} - a_0\,,\ a_r - a_0\,,$$

we obtain the formula

$$\psi(r\,,\,l\,;\,n) = c\,\{Q_n^{[1]}(a_1)\,,\ \dots,\ Q_n^{[1]}(a_{l-2})\,,\ Q_n^{[1]}(a_r)\} \tag{29.4}$$

where the later rows arise by replacing n by $n+1\,,\,n+2\,,\,\dots,\,n+l-2$. Here again $r \geqq l-1$. The constant factor c is positive and depends on l and n. We notice that $\psi_r(n)$ is now exhibited as a determinant of size $(l-1)\times(l-1)$ (instead of $l \times l$) expressed in terms of the orthogonal system $\{Q_n^{[1]}(x)\}$ whose spectrum consists of the values $a_1 < a_2 < a_3 < \dots$.

Repeating this procedure $l-1$ times we obtain

$$\psi(r\,,\,l\,;\,n) = c\,(l\,,\,n)\,Q_n^{[l]}(a_r) \tag{29.5}$$

where $c(l\,,\,n) > 0$; $r \geqq l-1$. Invoking the conclusions of Corollary 1 of Theorem 19 (Appendix, § 31) we have the following results:

(a) $\psi(r\,,\,l\,;\,x)$ has precisely $r-l+1$ sign changes as a sequence in n; $n = 0,\ 1,\ 2,\ \dots$.

(b) If we construct in the usual way the linear interpolation $\psi(r\,,l\,;x)$, $x \geqq 0$, these functions constitute for $r \geqq l-1$ a Sturm set.

§ 30. Miscellaneous remarks.

In this section we study a number of related topics of isolated character.

1. Hankel determinants formed from successive derivatives.

We investigate the distribution of the zeros of the polynomial defined by the following determinant of the Hankel type:

$$(30.1)\quad G_n(l\,;x) = G_n(x) = \begin{vmatrix} Q(x) & Q'(x) & \dots & Q^{(l-1)}(x) \\ Q'(x) & Q''(x) & \dots & Q^{(l)}(x) \\ \cdot & \cdot & \dots & \cdot \\ Q^{(l-1)}(x) & Q^{(l)}(x) & \dots & Q^{(2l-2)}(x) \end{vmatrix}$$

where $Q(x) = L_n^{(\alpha)}(x)$ is the Laguerre polynomial of degree n and of order α, $\alpha > -1$ and $Q^{(i)}$ designates now the i^{th} derivative of $Q(x)$. The normalization of $L_n^{(\alpha)}(x)$ will be clearly irrelevant. We follow the notation (5.11) so that $L_n^{(\alpha)}(-\infty) = +\infty$. We now prove the following

Theorem 16: Let $Q(x) = L_n^{(\alpha)}(x)$, $\alpha > -1$, and let $f(x)$ be any linear combination of the polynomial $Q(x)$ and its derivatives:

$$(30.2)\qquad f(x) = a_0\, Q(x) + a_1\, Q'(x) + \dots + a_{l-1}\, Q^{(l-1)}(x)$$

where the coefficients $a_0, a_1, \dots, a_{l-1}$ are real and not all zero. Then:

(a) $f(x)$ has at least $n-l+1$ nodal zeros contained in the open interval $(0, +\infty)$;

(b) $(-1)^{l(l-1)/2}\, G_n(l\,;x) > 0$ for all negative x, incl. $x = 0$;

(c) if l is even, then $(-1)^{l(l-1)/2}\, G_n(l\,;x) > 0$ for all real x;

(d) if l is odd, then $G_n(l\,;x)$, $n \geqq l-1$, has exactly $n-l+1$ simple zeros in $[0, \infty)$ and no other real zeros.

Proof of (a). We can assume that $1 \leqq l \leqq n$. We conclude (a) from the fact that if $R(x)$ denotes any polynomial of degree $n-l$ we have

$$(30.3)\qquad \int_0^\infty e^{-x} x^{\alpha+l-1} f(x)\, R(x)\, dx = 0\,,$$

showing that this equation is satisfied for every term of $f(x)$. Indeed, using the first part of (5.14) we have, $0 \leqq \nu \leqq l-1$,

$$\int_0^\infty e^{-x} x^{\alpha+l-1} Q^{(\nu)}(x) R(x)\, dx = \text{const.} \int_0^\infty e^{-x} x^{\alpha+\nu} L_{n-\nu}^{(\alpha+\nu)}(x) \cdot x^{l-1-\nu} R(x)\, dx = 0$$

since $x^{l-1-\nu} R(x)$ is of degree $n-\nu-1$.

Proof of (b) and (c). Suppose $G_n(l\,;x)$ vanishes at x_0, x_0 real. Then there exist some real constants a_i, $0 \leqq i \leqq l-1$, not all zero such that the linear combination $f(x)$ defined by (30.2) will have the form

$$f(x) = (x-x_0)^l R(x)$$

where $R(x)$ is a non-zero polynomial of degree not exceeding $n-l$. From (a) we know that the non-zero polynomial $f(x)$ has at least $n-l+1$ nodal zeros in $(0, \infty)$. Now, if $x_0 \leqq 0$, we can enumerate at least $n+1$ zeros (counting multiplicities) for $f(x)$. This obviously contradicts the fact that $f(x)$ is non-zero and of degree $\leqq n$. Thus we conclude that $G_n(l\,;x)$ never vanishes on the interval $[-\infty, 0]$.

Suppose now that $x_0 > 0$ but that l is even. Again we recall from (a) that $f(x)$ has at least $n-l+1$ nodal zeros in $(0, \infty)$. Since l is even, x_0 is either a zero of even multiplicity l or one of these same nodal zeros previously counted and then its multiplicity is at least $l+1$. In either case we deduce at least $n+1$ zeros, counting multiplicities, for $f(x)$ and we arrive at a contradiction as before. Hence $G_n(l\,;x)$ has no real zeros for l even as asserted.

The constant sign of $G_n(l\,;x)$ for $x \leqq 0$ and for all real x is easily obtained since the leading term of $G_n(l\,;x)$ is, apart from trivial positive factors,

$$(-1)^{ln+\frac{l(l-1)}{2}} \cdot x^{ln-\frac{l(l-1)}{2}}. \tag{30.4}$$

Proof of (d). Let l be odd; the proof proceeds by induction with respect to the degree n of $Q(x)$. For $l-1=n$ the determinant $G_n(l\,;x)$ is a constant since all derivatives beyond the l^{th} vanish identically while the derivative of order $l-1$ is constant. [We take the sign of $G_n(l\,;x)$

for large x into account.] We now propose to advance the induction step. Suppose we have proved that (30.1) has exactly $n-l+1$ nodal zeros in $(0, \infty)$ where $Q(x) = L_n^{(\alpha)}(x)$ for any $\alpha > -1$. Applying Sylvester's identity we have

$$\text{(30.5)} \quad \begin{vmatrix} G_{n+1}(l\,;x) & g_n(l\,;x) \\ G'_{n+1}(l\,;x) & g'_n(l\,;x) \end{vmatrix} = G_{n+1}(l+1\,;x)\, g_{n-1}(l-1\,;x) < 0$$

where $g_n(l\,;x)$, apart from a constant factor, coincides with $G_n(l\,;x)$ based on the polynomial

$$\frac{d}{dx} L_{n+1}^{(\alpha)}(x) = \text{const.}\; L_n^{(\alpha+1)}(x)\,;$$

G'_{n+1} and g'_n denote derivatives with respect to x. The inequality in (30.5) follows from (c) since $l+1$ and $l-1$ are even. Now, our induction hypothesis assures that g_n has exactly $n-l+1$ simple zeros in $(0, \infty)$. We conclude from (30.5) in the standard fashion that between two consecutive zeros of g_n there exists precisely one simple zeto of

$$G_{n+1}(l\,;x) = G_{n+1}$$

and similarly g_n possesses a zero between each two consecutive zeros of G_{n+1}. Moreover $(-1)^{l(l-1)/2} g'_n > 0$ at the largest zero of g_n and (30.5) then implies that $(-1)^{l(l-1)/2} G_{n+1}$ is negative at this point. Since $(-1)^{l(l-1)/2} G_{n+1}$ is positive for large x we conclude the existence of a single zero for G_{n+1} which is larger than all the zeros of g_n. In a similar way we establish the existence of a positive zero for G_{n+1} smaller than each zero of g_n. We have thus demonstrated the validity of our result for the function $G_{n+1}(l\,;x)$. The induction step is complete, and the proof of the Theorem is hereby finished.

The function $G_n(l\,;x)$ as in (30.1) based on the Hermite polynomial $H_n(x)$ reduces to the Wronskian $W(H_n(x), H_{n+1}(x), \ldots, H_{n+l-1}(x))$. In this case the oscillation characteristics of $G_n(l\,;x)$ follow from Theorems 1 and 2.

The nature of the zeros of $G_n(l\,;x)$ where $Q(x) = P_n^{(\lambda)}(x)$, the ultraspherical polynomial, remains an open problem. We can prove the analog of the Theorem only when $l \leq 4$.

An analogous reasoning leads to the following theorem on Meixner polynomials where we follow the notation of § 5, § 20, § 22:

Theorem 17: Let $a_0, a_1, \ldots, a_{l-1}$ be arbitrary real and not all zero, $n \geqq l-1$. The polynomial, $\beta > 0$, $0 < \gamma < 1$,

$$f(x) = \sum_{i=0}^{l-1} a_i M_{n-i}(\beta + i\,;\, x), \qquad M_n(\beta\,, \gamma\,;\, x) = M_n(\beta\,;\, x),$$

has at least $n - l + 1$ nodal zeros located in the interval $(0\,, \infty)$.

The proof is based on the fact that $f(x)$ is orthogonal to any polynomial of degree not exceeding $n - l$ with respect to the measure which places mass

$$j(x) = (1-\gamma)^{\beta+l-1} \frac{(\beta + l - 1)_x \gamma^x}{x\,!} \qquad \text{at} \qquad x = 0,\ 1,\ 2,\ \ldots .$$

The necessary details are in fact worked out already in § 22, cf. the argument following (22.10) where x must be replaced by n, and $l - 1 - \mu$ by i.

2. Jacobi polynomials.

We use the notation $P_n^{(\alpha\,,\,\beta)}(x)$ of [13, p. 58] for the general Jacobi polynomials. In § 3.3 it was shown that Turán's inequality ($l = 2$) is not valid in general for the polynomials

$$Q_n(\alpha\,, \beta\,;\, x) = \frac{P_n^{(\alpha\,,\,\beta)}(x)}{\binom{n+\alpha}{n}}\,, \qquad Q_n(\alpha\,, \beta\,;\, 1) = 1\,. \tag{30.6}$$

We prove now the following modified assertion.

Theorem 18: Let α be fixed, $\alpha > -1$; we write

$$Q_n(\alpha\,, \beta\,;\, x) = Q_n(\beta\,;\, x)\,. \tag{30.7}$$

The following inequality holds for $\beta > 0$ and for all real values of x, $x \neq 1$:

$$[Q_n(\beta\,;\, x)]^2 - Q_{n-1}(\beta + 1\,;\, x)\, Q_{n+1}(\beta - 1\,;\, x) > 0\,. \tag{30.8}$$

Let N be a positive integer, b and c real. We have the following identity [2, Vol. 1, p. 85].

$$(30.9)\quad \sum_{n=0}^{N}\binom{N}{n}z^n F(-n, b; c; u) = (1+z)^N F\left(-N, b; c; \frac{uz}{1+z}\right).$$

Each term is a polynomial in u. Moreover we use the identity [13, (4.3.2)]

$$(30.10)\quad Q_n(\beta; x) = \frac{P_n^{(\alpha,\beta)}(x)}{\binom{n+\alpha}{n}} = \left(\frac{x+1}{2}\right)^n F\left(-n, -n-\beta; \alpha+1; \frac{x-1}{x+1}\right).$$

We choose in (30.9): $b=-N-\beta$, $c=\alpha+1$. Since

$$\begin{aligned} F(-n, b; c; u) &= F(-n, -n-(N-n+\beta); \alpha+1; u) \\ &= \left(\frac{2}{x+1}\right)^n Q_n(N-n+\beta; x), \quad u=\frac{x-1}{x+1}, \end{aligned}$$

we have the following identity:

$$\sum_{n=0}^{N}\binom{N}{n}\left(\frac{2z}{x+1}\right)^n Q_n(N-n+\beta; x) = (1+z)^N F\left(-N, -N-\beta; \alpha+1; \frac{uz}{1+z}\right),$$

$$u = \frac{x-1}{x+1}.$$

Let x be real, $x \neq \pm 1$ so that u is finite and $\neq 0$. The right hand expression can be written, in view of (30.10), in the form

$$\left(\frac{x+1+2z}{x+1}\right)^N Q_N\left(\beta; \frac{x+1+2xz}{x+1+2z}\right).$$

As a polynomial in z it is of degree N and has only real zeros so that (§ 1.7) the inequality (30.8) follows.

The case $x=-1$ requires only a slight modification, and the result is the same.

For $\beta \to +\infty$ we obtain [cf. (5.36)] Turán's inequality for the Laguerre polynomials, $l=2$, for all real x, however in the weak form ($\geqq 0$ instead of >0).

3. Limit relations.

From the inequalities proved in the course of this investigation we may be able to derive further inequalities by using appropriate limit relations.

(a) We show that the higher order Turán inequality for the Laguerre polynomial system

$$Q_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) \,/\, L_n^{(\alpha)}(0),$$

cf. (5.16), emerges, at least in the weaker form ($\geqq 0$ instead of >0) as a limiting case of the higher order Turán inequality applied to the Meixner polynomials (Theorem 8). For this purpose we refer to the relation (5.35). Let x be a fixed positive value. We choose γ approaching 1 and N a positive integer increasing to infinity in such a way that the quotient $\dfrac{(1-\gamma)N}{\gamma}$ has the fixed value x. We use the inequality of Theorem 8 (c) at the value N, i.e.

$$(-1)^{l/2}\, T(M_n(N)\,,\, M_{n+1}(N)\,,\, \ldots\,,\, M_{n+l-1}(N)) > 0\,, \qquad l \text{ even},$$

and then proceed to the limit as directed by (5.35). The result obtained is the following weaker form of the higher order Turán inequality for the Laguerre system:

$$(-1)^{l/2}\, T\,(Q_n^{(\alpha)}(x)\,,\, Q_{n+1}^{(\alpha)}(x)\,,\, \ldots\,,\, Q_{n+l-1}^{(\alpha)}(x)) \geqq 0$$

where $\alpha > -1$ is arbitrary.

(b) An appropriate limit operation performed on the Laguerre polynomials produces the Bessel function. Specifically [13, (8.1.8)]

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha)}\left(\dfrac{x}{n}\right)}{\Gamma(\alpha+1)} = x^{-\frac{\alpha}{2}}\, J_\alpha(2\sqrt{x}) \tag{30.11}$$

where $Q_n^{(\alpha)}(x)$ has the meaning (5.16) and J_α follows the standard notation [13, (1.71.1)]. Using the higher order Turán inequality for the Laguerre system as well as the identity (11.8), cf. Remark, we obtain that for $x>0$, $\alpha>-1$, l even,

$$(-1)^{l/2} \left| \frac{Q_n^{(\alpha+\mu+\nu)}\left(\frac{x}{n}\right)}{\Gamma(\alpha+\mu+\nu+1)} \right|_0^{l-1} > 0\,.$$

So we deduce with the aid of (30.11),

(30.12)

$$(-1)^{\frac{l}{2}}\, T\,(x^{-\frac{\alpha}{2}} J_\alpha(2\sqrt{x})\,,\, x^{-\frac{\alpha+1}{2}} J_{\alpha+1}(2\sqrt{x}),\, \ldots,\, x^{-\frac{\alpha+l-1}{2}} J_{\alpha+l-1}(2\sqrt{x})) \geqq 0\,.$$

The functions indicated appear in the first row; the later rows arise by replacing α by $\alpha+1\,,\, \alpha+2\,,\, \ldots,\, \alpha+l-1\,,$ respectively. Inspection of this

determinant instantly reveals that we may cancel out factors of powers of x thus securing

$$(30.13) \qquad (-1)^{l/2}\, T\,(J_\alpha(2\sqrt{x})\,,\, J_{\alpha+1}(2\sqrt{x})\,,\,\dots,\, J_{\alpha+l-1}(2\sqrt{x})) \geqq 0\,.$$

Undoubtedly strict inequality prevails in (30.13) but we leave the study of this problem to another opportunity when we shall also examine the Wronski and Turán inequalities for general Sturm-Liouville systems [cf. Szász 12].

Using (30.11) again with reference to § 30.1 (c), we obtain the further inequality

$$(30.14) \qquad (-1)^{l/2}\,[D^{\mu+\nu}\,\varphi_\alpha(x)]_0^{l-1} \geqq 0\,, \qquad D = \frac{d}{dx}\,,$$

where $\varphi_\alpha(t) = t^{-\alpha/2}\, J_\alpha(2\sqrt{t})$ and l is even; $x > 0$, $\alpha > -1$.

4. Alternative normalizations in Turán's inequality.

What are the relevant normalizations under which the inequality of the Turán type ($l = 2$) holds? This question is generally unresolved as yet. The following remark is helpful.

Let $\{Q_n(x)\}$ denote a system of polynomials for which the inequality of Turán ($l = 2$) has been established. Let $\{\alpha_n\}$ denote any sequence of positive constants obeying the inequalities

$$(30.15) \qquad \begin{vmatrix} \alpha_{n-1} & \alpha_n \\ \alpha_n & \alpha_{n+1} \end{vmatrix} \leqq 0\,, \qquad n = 1, 2, 3, \dots\,.$$

Then $P_n(x) = \alpha_n\, Q_n(x)$ also satisfies Turán's inequality. Indeed with obvious abbreviation,

$$(30.16) \qquad \begin{aligned} P_n^2 - P_{n-1}\, P_{n+1} &= \alpha_n^2\, Q_n^2 - \alpha_{n-1}\,\alpha_{n+1}\, Q_{n-1}\, Q_{n+1} \\ &= (\alpha_n^2 - \alpha_{n-1}\,\alpha_{n+1})\, Q_n^2 + \alpha_{n-1}\,\alpha_{n+1}\,(Q_n^2 - Q_{n-1}\, Q_{n+1}) \geqq 0\,. \end{aligned}$$

Let us indicate a few simple applications of (30.16).

(a) We consider the ultraspherical polynomials $P_n^{(\lambda)}(x)$ in the standard notation (5.3). We choose

$$\alpha_n = P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n},$$

cf. (5.4), which satisfies (30.15) for $\lambda \geqq \frac{1}{2}$. On account of the validity of Turán's inequality for

$$Q_n(x) = P_n^{(\lambda)}(x) / P_n^{(\lambda)}(1)$$

we deduce the Turán inequality for $P_n^{(\lambda)}(x)$ when $\lambda \geqq \frac{1}{2}$. Direct calculation shows that the conclusion is false for $\lambda < \frac{1}{2}$.

(b) The Laguerre polynomials $L_n^{(\alpha)}(x)$ as defined by (5.11) obey the Turán inequality provided $\alpha \geqq 0$. This is verified as in (a) using the fact that

$$Q_n(x) = L_n^{(\alpha)}(x) / L_n^{(\alpha)}(0)$$

satisfies Turán's inequality and that

$$\alpha_n = L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$$

satisfies (30.15).

(c) Let $\{Q_k(x)\,;\, k = 0, 1, \ldots, N\}$ be a finite system of orthogonal polynomials with a discrete spectrum S such that for x on S

$$\sum_{k=0}^{N} \binom{N}{k} Q_k(x)\, z^k$$

represents a polynomial in z (generating function) with only real zeros. Then we know that Turán's inequality holds for $Q_k(x)$ provided x is in S. It follows further that $Q_k(x) / (N-k)!$ also satisfies a Turán inequality.

5. Charlier-Poisson polynomials: Turán's inequality is not valid outside of the spectrum.

We follow the notation (5.18) for these polynomials

$$c_n(a\,;\, x) = c_n(x)$$

and show that if x is positive and non-integral, the determinant of the Turán type

$$(30.17)\quad T(c_n(x)\,,\, c_{n+1}(x)) = \begin{vmatrix} c_n(x) & c_{n+1}(x) \\ c_{n+1}(x) & c_{n+2}(x) \end{vmatrix} = -\frac{x}{a} \begin{vmatrix} c_n(x-1) & c_n(x) \\ c_{n+1}(x-1) & c_{n+1}(x) \end{vmatrix}$$

has the "wrong" sign, i.e. positive for n sufficiently large. For this fact we give two different proofs. [For the second representation of T, cf. (22.1).]

(a) Let $k < x < k+1$; we first observe that $c_n(x)$ has a single sign $(-1)^{k+1}$ if n is sufficiently large; also it increases in magnitude to infinity at a rate faster than any fixed power of n. To see this, we decompose the explicit formula (5.18) for $c_n(x)$ into two parts:

$$c_n(x) = \sum_{\nu=0}^{k+1} (-1)^\nu \binom{n}{\nu} \binom{x}{\nu} \frac{\nu!}{a^\nu} + (-1)^{k+1} \tag{30.18}$$

$$\cdot \sum_{\nu=k+2}^{\infty} \binom{n}{\nu} \frac{1}{a^\nu} x(x-1) \dots (x-\nu) \cdot (\nu+1-x)(\nu+2-x) \dots (k-1-x)$$

$$= I_1 + I_2$$

where the terms of I_2 are obviously all of the same constant sign. Examination of each term clearly implies that I_1 increases slower than the power n^{k+1} while I_2 increases faster than any fixed power of n, so that $c_n(x)$ ultimately achieves the fixed sign $(-1)^{k+1}$. Furthermore, we verify that for $k < x < k+1$

$$(-1)^{k+1} \sum_{n=0}^{\infty} \frac{a^n}{n!} c_n(x) n^{k+1} = +\infty . \tag{30.19}$$

We begin with the generating function (20.1) valid for $|z| < a$. Since $c_n(x)$ is ultimately of a single sign we can apply Abel's lemma and let $z \to a - 0$ in (20.1). The resulting series converges absolutely and we have clearly

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} c_n(x) = 0 .$$

Differentiation of the generating function l times, $l \leqq k$, and appeal to Abel's lemma again, yields

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} |c_n(x)| n^l < \infty .$$

However differentiating $k+1$ times we obtain by Abel's lemma an absolutely divergent series

$$(-1)^{k+1} \sum_{n=0}^{\infty} \frac{a^n}{n!} c_n(x) \cdot n(n-1) \dots (n-k) = +\infty . \tag{30.20}$$

Thus, since $|c_n(y)| > n^{k+1}$ for y a non-integer value say $l < y < l+1$, we conclude on the basis of (30.20) that

$$\lim_{n \to \infty} (-1)^{k+l} \sum_{\nu=0}^{n} \frac{a^\nu}{\nu!} c_\nu(x) c_\nu(y) = +\infty . \tag{30.21}$$

We are prepared now to verify the non-validity of the Turán inequality ($l = 2$) for the Poisson-Charlier polynomials when x is non-integer. Comparing the right hand side of (30.17) with the Christoffel-Darboux formula [13, (3.2.3)] we find

$$\frac{a^{n+1}}{n!} T(c_n(x), c_{n+1}(x)) = -\frac{x}{a} \sum_{\nu=0}^{n} \frac{a^\nu}{\nu!} c_\nu(x) c_\nu(x-1). \tag{30.22}$$

When $k < x < k+1$, (30.21) implies that

$$\lim_{n\to\infty} \sum_{\nu=0}^{n} \frac{a^\nu}{\nu!} c_\nu(x) c_\nu(x-1) = -\infty$$

and thus $T(c_n(x), c_{n+1}(x)) > 0$ for n sufficiently large as claimed.

(b) Let x be positive and non integer. The generating function (20.1) has then a branch point at $z = a$. We apply Darboux's method [13, pp. 204—206]. By inserting in (20.1)

$$e^z = e^a \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} (1 - a^{-1} z)^k$$

and expanding the single terms, we obtain the following asymptotic expansion, $n \to \infty$,

$$\frac{c_n(x)}{n!} \sim e^a \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} (-1)^n a^{-n} \binom{x+k}{n};$$

in particular,

$$\frac{c_n(x)}{n!} = e^a \cdot (-1)^n a^{-n} \binom{x}{n} \left\{1 + \frac{b}{n} + O\left(\frac{1}{n^2}\right)\right\}$$

where b is a real constant independent of n. Hence

$$c_n(x) = e^a \cdot (-1)^n a^{-n} \frac{\Gamma(x+1)}{\Gamma(x+1-n)} \left\{1 + \frac{b}{n} + O\left(\frac{1}{n^2}\right)\right\} \tag{30.23}$$

so that

$$\frac{c_n(x)}{c_{n-1}(x)} = -\frac{x-n+1}{a} \left\{1 + O\left(\frac{1}{n^2}\right)\right\},$$

$$T(c_{n-1}(x), c_n(x)) = [c_n(x)]^2 \left\{\frac{c_{n-1}(x)}{c_n(x)} \cdot \frac{c_{n+1}(x)}{c_n(x)} - 1\right\}$$

$$= \left\{e^a \cdot (-1)^n a^{-n} \frac{\Gamma(x+1)}{\Gamma(x+1-n)}\right\}^2 \left\{1 + \frac{b}{n} + O\left(\frac{1}{n^2}\right)\right\}^2 \left\{\frac{n-x}{n-1-x} - 1 + O\left(\frac{1}{n^2}\right)\right\}.$$

The expression occuring in the last bracket is

$$\frac{1}{n} + O\left(\frac{1}{n^2}\right);$$

consequently,

$$\lim_{n\to\infty}\left(\frac{\Gamma(x+1-n)}{\Gamma(x+1)}\right)^2 \cdot na^{2n} \cdot T(c_{n-1}(x), c_n(x)) = e^{2a}.$$

Hence T is positive for sufficiently large n.

6. Charlier-Poisson polynomials: The associated Jensen polynomials.

It is instructive to evaluate the Jensen polynomials $f_N(z)$ associated with the generating function (20.1) of the Poisson-Charlier polynomials and discuss the reality of their zeros. We have

$$(30.24)\quad f_N(z) = \sum_{n=0}^{N}\binom{N}{n} c_n(x) z^n = \sum \binom{N}{n}(-1)^\nu \binom{n}{\nu}\binom{x}{\nu}\frac{\nu!}{a^\nu} z^n.$$

Here we used the explicit representation (5.18); the letters of summation are subjected to the condition $0 \leqq \nu \leqq n \leqq N$.

Performing first the summation with respect to n we find

$$\begin{aligned}\sum_{n=\nu}^{N}\binom{N}{n}\binom{n}{\nu} z^n &= \frac{N!}{(N-\nu)!\,\nu!}\sum_{n=\nu}^{N}\binom{N-\nu}{N-n} z^n \\ &= \frac{N!}{(N-\nu)!\,\nu!} z^\nu (1+z)^{N-\nu}\end{aligned}$$

so that

$$\begin{aligned} f_N(z) &= \sum_{\nu=0}^{N}\frac{N!}{(N-\nu)!} z^\nu (1+z)^{N-\nu} \cdot (-1)^\nu \binom{x}{\nu}\frac{1}{a^\nu} \\ (30.25)\qquad &= N!\sum_{\nu=0}^{N}\binom{x}{N-\nu}\frac{(-1)^{N-\nu}}{\nu!}\left(\frac{z}{a}\right)^{N-\nu}(1+z)^\nu \\ &= N!\left(-\frac{z}{a}\right)^N L_N^{(x-N)}\left(\frac{1+z}{z}a\right).\end{aligned}$$

Let x be positive, N a positive integer. If $N < x+1$, the zeros of $f_N(z)$ are all real. If $N > x+1$ and x is an integer, the zeros are still

real [cf. 13, (5.2.1)]; in these cases Turán's inequality holds. If $N > x+1$ but x is not integer, the function $f_N(z)$ will have $N-2-[x]$ or $N-1-[x]$ complex zeros according as $N-[x]$ is even or odd [cf. 13, Theorem 6.73].

7. Application of Theorem 3 to finite systems.

The theory of the reduced moment problem in combination with orthogonal polynomials permits to obtain various inequalities of some interest by suitable application of Theorem 3. Let $\{Q_n(x)\}$ represent an arbitrary infinite system of orthogonal polynomials normalized for convenience so that $Q_n(-\infty) = +\infty$. Let the integer N be fixed and denote by $x_0 < x_1 < x_2 < \ldots < x_N$ the zeros of the "quasi-orthogonal polynomial" $\lambda Q_N(x) + \mu Q_{N+1}(x)$ where λ and μ are arbitrary fixed real constants. It is a familiar fact then that there exists a discrete measure ψ_N concentrating its masses fully on the values x_i such that the finite system

$$\{Q_i(x)\,;\, i = 0, 1, \ldots, N-1\}$$

is orthogonal with respect to ψ_N. Invoking Theorem 3 we conclude that

$$(-1)^{l/2}\, Q\begin{pmatrix} n\,, & n+1\,, & \ldots, & n+l-1 \\ x_k\,, & x_{k+1}\,, & \ldots, & x_{k+l-1} \end{pmatrix} > 0$$

provided $n + l \leqq N$, $k + l + 1 \leqq N$ and that l is even.

8. A counterpart of Theorem 10.

We consider the ultraspherical polynomials

$$Q_n(x) = P_n^{(\lambda)}(x) \,/\, P_n^{(\lambda)}(1)$$

so that

$$Q_n(1) = 1\,;\quad \lambda > -\frac{1}{2}\,.$$

Let k be a fixed non negative integer. As a complement of Theorem 10 we prove that the extended "augmented" system

$$\psi_n(k\,;x) = \psi_n(x) = \begin{vmatrix} Q_k(x) & Q_n(x) \\ Q_{k+1}(x) & Q_{n+1}(x) \end{vmatrix}, \tag{30.26}$$

$$n = k-1\,,\, k-2\,,\, \ldots,\, 1,\, 0,\, -1,\, -2,\, \ldots\,,$$

constitutes also a Sturm set provided we accept the convention

$$Q_{-n}(x) = Q_{n-1}(x)\,.$$

(In the case of the Legendre polynomials, replacing n by $-n-1$, the differential equation (5.5) does not change.)

For the proof we use the identities [(5.6); 13, (4.7.27)]

$$(30.27) \qquad \begin{aligned}(1-x^2)\,Q'_n(x) &= n\,[Q_{n-1}(x) - x\,Q_n(x)] \\ &= (n+2\lambda)\,[x\,Q_n(x) - Q_{n+1}(x)]\,, \qquad n \geqq 0\,.\end{aligned}$$

(In the Legendre case the two formulas interchange if n is replaced by $-n-1$.) We have to verify the properties of § 2. For $1 \leqq n \leqq k-1$ the proof is the same as for $n > k$, cf. § 24, so that we consider only the cases $n = -m$, $m = 3, 4, \ldots$, and in addition the cases $n = 0, -1, -2$.

We have

$$(30.28) \qquad \psi_{-m}(x) = \begin{vmatrix} Q_k(x) & Q_{m-1}(x) \\ Q_{k+1}(x) & Q_{m-2}(x) \end{vmatrix} = \begin{vmatrix} Q_k(x) & Q_{m-1}(x) \\ \dfrac{-(1-x^2)\,Q'_k(x)}{k+2\lambda} & \dfrac{(1-x^2)\,Q'_{m-1}(x)}{m-1} \end{vmatrix}$$

the last identity resulting by virtue of (30.27). Let x_0 be a zero of $\psi_{-m}(x)$ in $-1 < x_0 < 1$. Then a constant t exists, $t \neq 0$ (see the argument of § 24) such that

$$(30.29) \qquad Q_k(x_0) = t\,Q_{m-1}(x_0)\,, \quad Q_{k+1}(x_0) = t\,Q_{m-2}(x_0)\,.$$

Moreover we have

$$(30.30) \qquad Q'_k(x_0) = -\frac{k+2\lambda}{m-1}\,t\,Q'_{m-1}(x_0)\,.$$

Differentiation of the second expression (30.28) with reference to the differential equation (5.5), to $\psi_{-m}(x_0) = 0$ and to (30.29), (30.30) leads, after a rather straight forward calculation, to the expression

$$(30.31) \qquad \psi'_{-m}(x_0) = -t\,(m+k+2\lambda-1)\left\{[Q_{m-1}(x_0)]^2 + \frac{(1-x_0^2)\,[Q'_{m-1}(x_0)]^2}{(m-1)^2}\right\}.$$

Similarly, we obtain

$$\psi_{-m-1}(x_0) = t\begin{vmatrix} Q_{m-1}(x_0) & Q_m(x_0) \\ Q_{m-2}(x_0) & Q_{m-1}(x_0) \end{vmatrix}, \quad \psi_{-m+1}(x_0) = t\begin{vmatrix} Q_{m-1}(x_0) & Q_{m-2}(x_0) \\ Q_{m-2}(x_0) & Q_{m-3}(x_0) \end{vmatrix}$$

for any zero x_0 of $\psi_{-m}(x)$ located in the open interval $(-1\,,\,1)$. Thus

$$(30.32) \qquad \psi'_{-m}(x_0)\,\psi_{-m-1}(x_0) < 0\,,$$

$$(30.33) \qquad \psi'_{-m}(x_0)\,\psi_{-m+1}(x_0) > 0\,.$$

There is no difficulty in settling the remaining cases:

(i) $n=0$, $Q_k=t$, $Q_{k+1}=tx_0$, $\psi_{-1}=t(1-x_0)$, $\psi_1=t(Q_2-Q_1^2)$;

(30.33) follows from § 24, moreover $\operatorname{sgn}\psi_{-1}=-\operatorname{sgn}\psi_1$;

(ii) $n=-1$; $Q_k=Q_{k+1}=t$, $\psi_0=-\psi_{-2}=t(x_0-1)$, $\operatorname{sgn}\psi'_{-1}=-\operatorname{sgn}t$;

(iii) $n=-2$; $Q_k=tx_0$, $Q_{k+1}=t$, $\psi_{-1}=t(x_0-1)$, $\psi_{-3}=t(Q_1^2-Q_2)$, $\operatorname{sgn}\psi'_{-2}=-\operatorname{sgn}t$.

APPENDIX

§ 31. Oscillation properties of the zeros of $Q_n(x)$.

1. The duality explained in § 1 and stressed several times during the course of this investigation, suggests a further detailed discussion of the interplay between the variables x and n with particular reference to the oscillation properties of the orthogonal polynomials $Q_n(x)$. The classical theory of orthogonal polynomials tells us that $Q_n(x)$ has exactly n simple zeros and the zeros of two successive functions interlace. Also, the initial function $Q_0(x)$ is of one sign. In this point of view the index variable n generates the successive functions. In a parallel manner it is possible to consider various choices of x and thus generate successive functions of n through the identification $R_x(n)=Q_n(x)$.

In developing a dual theory pertaining to the zeros of the family of functions $R_x(n)$, it is generally necessary to restrict the values of x to the spectrum of the given measure ψ.

In a certain sense the functions $R_x(n)$ arising for different choices of x, x on the spectrum, are orthogonal with respect to a measure π_n (cf. (25.2)) at n; here

$$\pi_n^{-1}=\int [Q_n(x)]^2\,d\psi(x)$$

and $Q_n(x)$ is normalized appropriately. In an appropriate sense we have the relation

$$\sum_{n=0}^{\infty}\pi_n R_x(n)\,R_y(n)\sim\delta_{x,y}\cdot k(x) \tag{31.1}$$

where $\delta_{x,y}$ denotes the standard δ function.

If x and y are distinct mass points of the measure ψ, the series (31.1) actually converges (in fact absolutely) and equals to zero. When $x=y$, the value $k(x)$ is the reciprocal of the mass placed by ψ at x. The proof of these assertions follows by verifying that $R_x(n)$, when x is a value on the point spectrum of ψ, represents an eigenfunction of a suitable self-adjoint operator defined on a Hilbert space whose elements are composed of sequences. The expression (31.1) represents the scalar product in the Hilbert space evaluated for the eigenfunctions $R_x(n)$ and $R_y(n)$. The orthogonality of eigenfunctions corresponding to distinct eigenvalues implies (31.1). The formal details of the proof of these statements are omitted. A full discussion of the moment problem and its relation to orthogonal polynomials from this point of view, can be found in the treatise of M. H. Stone.[11]

2. We consider now the case when the spectrum of the measure ψ is totally discrete with certain positive masses located at a_r, $r=0, 1, 2, \ldots$; $a_0 < a_1 < a_2 < \ldots$. In a natural way we then form the functions of n:

$$R_{a_0}(n)\ ,\ R_{a_1}(n)\ ,\ R_{a_2}(n), \ldots \tag{31.2}$$

as well as their linear interpolation $R_{a_r}(z)$ defined in the usual way [cf. § 2]. More generally we consider also the functions $R_a(n)$ and $R_a(z)$ where a is arbitrary.

We make the following important distinction.[12]

(i) The moment problem associated with the measure ψ is uniquely determined. In this case the series $\sum_{n=0}^{\infty} \pi_n |Q_n(x)|^2$ converges if and only if x is a mass point of the spectrum of ψ.

(ii) The moment problem associated with the measure is not unique. In this case $\lim_{n\to\infty} Q_n(x) = Q(x)$ exists for all x provided we adopt the normalization $Q_n(a_0)=1$, and $Q(x)$ is an entire function whose zeros are all simple and coincide with the spectral set $(a_0, a_1, a_2, \ldots)$.

11. M. H. Stone, Linear transformations in Hilbert space and their applications to Analysis, American Mathematical Society, Colloquium Publications, Vol. 15, 1932; cf. Chapter 7.

12. J. A. Shohat and J. D. Tamarkin, The problem of moments, Mathematical surveys, Nr. 1, 1943; cf. p. 50.

3. Our next objective is the proof of

Theorem 19: *Let ψ be the discrete measure defined in* **2**, *and let us denote by $\{Q_n(x)\}$ the associated orthogonal polynomials.*

(a) Each polynomial $Q_n(x)$ can have at most one zero in the closed interval $I_r = [a_r, a_{r+1}]$.

(b) Let $a_r < \xi \leqq a_{r+1}$. There exists a number $n(r, \xi)$ such that $Q_n(x)$ vanishes exactly once in the interval $T_r = (a_r, \xi]$ for all $n \geqq n(r, \xi)$.

Part (a) of the assertion is clear. Indeed, suppose to the contrary that two successive zeros x', x'' of $Q_n(x)$ lie in I_r. We construct a polynomial $h(x)$ of degree $n-2$ vanishing at all the zeros of $Q_n(x)$ except at x' and x''. It is easy to see that the non-zero polynomial $Q_n(x)\, h(x)$ does not change sign at the points a_k. But $Q_n(x)$ is orthogonal to $h(x)$ which is impossible.

Part (b) of the assertion is simple provided that the case (ii) occurs. Indeed, by Hurwitz's theorem the zeros of $Q_n(x)$ must converge for $n \to \infty$ to the spectral points a_r. The first zero tends to a_0 from the right so that the second zero can not be $\leqq a_1$, if n is large enough, hence it is $> a_1$ for large n; similarly the next zero must be $> a_2$ for large n; and so on.

Now we assume that the case (i) occurs, i.e. that the moment problem is unique. The proof of (b) will be carried out by induction with respect to r. We suppose, for the sake of simplicity that $a_0 = 0$ and $Q_n(0) = 1$. Hence the coefficient of the highest power in $Q_n(x)$ is of the sign $(-1)^n$.

First we observe [cf. § 29.2] that the polynomials

$$Q_n^{[1]}(x) = -x^{-1}\,[Q_{n+1}(x) - Q_n(x)]$$

are orthogonal with respect to the measure $d\alpha(x) = x\, d\psi(x)$; the spectral points of this new measure are the same as those of ψ except that $a_0 = 0$ has been discarded. Since $Q_n^{[1]}(x) > 0$ for $x \leqq a_1$, we have $Q_{n+1}(x) < Q_n(x)$ for $0 < x \leqq a_1$.

Suppose that $Q_n(x)$ does not vanish in $T_0 = (0, \xi]$, $0 < \xi \leqq a_1$ for infinitely many n. Then, by virtue of the normalization condition at $x = 0$, $Q_n(x)$ as a function of x decreases in this interval and consequently

$$0 < Q_n(\xi) < Q_n(0) = 1\,.$$

This inequality holds for certain arbitrarily large, hence for all sufficiently large values of n since $Q_{n+1}(\xi) < Q_n(\xi)$. Thus

$$\sum \pi_n [Q_n(\xi)]^2 < \sum \pi_n [Q_n(0)]^2 = \sum \pi_n$$

which implies that ξ is a point on the spectrum of ψ contrary to the hypothesis.

We proceed to the general induction step. Suppose the assertion (b) has been verified in the case of the r^{th} interval T_r for any system of orthogonal polynomials whose spectral measure is discrete and located on the non-negative real line. We shall establish the result in the case of

$$T_{r+1} = (a_{r+1}, \xi]$$

where

$$a_{r+1} < \xi \leqq a_{r+2}.$$

By virtue of the induction hypothesis the polynomials $Q_n^{[1]}(x)$ for sufficiently large n will have the sign $(-1)^r$ when $x = a_{r+1}$ and the opposite sign for $x = \xi$. Thus

$$(31.3) \qquad \begin{cases} (-1)^{r+1} Q_{n+1}(a_{r+1}) > (-1)^{r+1} Q_n(a_{r+1}), \\ (-1)^{r+1} Q_{n+1}(\xi) < (-1)^{r+1} Q_n(\xi) \quad \text{for all } n \geqq n(r, \xi). \end{cases}$$

Moreover, appealing to (a) and the induction hypothesis we infer for all sufficiently large n that $(-1)^{r+1} Q_n(a_{r+1}) > 0$.

Now let us assume that

$$(-1)^{r+1} Q_n(\xi) \geqq 0,$$

hence

$$(-1)^{r+1} Q_n(x) \geqq 0 \ , \ a_{r+1} \leqq x \leqq \xi,$$

for infinitely many n, say for

$$n_0 < n_1 < n_2 < \ldots, \qquad n_0 \geqq n(r, \xi).$$

With the aid of the second part of (31.3) we see that the sequence $(-1)^{r+1} Q_n(\xi)$ is decreasing in n and has positive terms for $n \geqq n_0$. Hence $|Q_n(\xi)|$ is bounded and since $\Sigma \pi_n$ is convergent, the same will hold for $\Sigma \pi_n [Q_n(\xi)]^2$ which is impossible. Thus $(-1)^{r+1} Q_n(\xi) < 0$ for sufficiently large n.

This establishes the proof of Theorem 19.

Corollary: Let $\{Q_n(x)\}$ have the same meaning as in Theorem 19. For fixed r and for sufficiently large n the following inequality holds:

$$\frac{Q_n(a_r)}{Q_{n+1}(a_r)} > \frac{Q_n(a_{r+1})}{Q_{n+1}(a_{r+1})} . \tag{31.4}$$

According to Theorem 3:

$$Q\begin{pmatrix} n\ , & n+1 \\ a_r\ , & a_{r+1} \end{pmatrix} = \begin{vmatrix} Q_n(a_r) & Q_{n+1}(a_r) \\ Q_n(a_{r+1}) & Q_{n+1}(a_{r+1}) \end{vmatrix} < 0$$

and for large n:

$$\operatorname{sgn} Q_{n+1}(a_r)\, Q_{n+1}(a_{r+1}) = (-1)^r \cdot (-1)^{r+1} = -1 .$$

4. We prove further

Theorem 20: Let a_r, $\{Q_n(x)\}$, $R_a(z)$ have the same meaning as before.

(i) If $a_r < a \leqq a_{r+1}$, then $R_a(z)$ has precisely $r+1$ nodal zeros for $z>0$.

(ii) If $a_r < b < c \leqq a_{r+1}$, the zeros of $R_b(z)$ and $R_c(z)$ strictly interlace.

Proof of (i).

Let $a = a_r$ (or a_{r+1}); assertion (i) follows from § 10.3 by the specialization $l = 1$ since

$$u_n(r) = Q_n(a_r)\ ,\ u_z(r) = R_{a_r}(z) .$$

If a is arbitrary, $a_r < a \leqq a_{r+1}$ we apply the familiar fact that the polynomials $\{Q_n(x)\}$ form a Sturm sequence in the conventional sense[13] in every positive interval; more specifically, assuming that $Q_n(a) \neq 0$, the number of sign changes in the sequence

$$Q_0(a)\ ,\ Q_1(a),\ \ldots,\ Q_n(a) \tag{31.5}$$

coincides with the number of zeros of $Q_n(x)$ in the interval $(0\ ,\ a)$. Hence

13. Cf. for instance, J. V. Uspensky, Theory of equations, 1948, Chapter VII, pp. 138—150.

(i) follows immediately by taking n so large that $\operatorname{sgn} Q_m(a_r) = (-1)^r$, and $\operatorname{sgn} Q_m(a) = (-1)^{r+1}$ for $m \geqq n$.

Proof of (ii).

We denote by $z_i(b)$ and $z_i(c)$; $i = 1, 2, \ldots, r+1$, the nodal zeros of $R_b(z)$ and $R_c(z)$, respectively, and show that

(31.6) $\quad 0 < z_1(c) < z_1(b) < z_2(c) < z_2(b) < \ldots < z_{r+1}(c) < z_{r+1}(b)$.

First we point out that $z_i(b)$ is a continuous and strictly decreasing function of b. Indeed, if

$$z_i(b) = \rho n + (1-\rho)(n+1), \qquad 0 \leqq \rho \leqq 1,$$

we have

$$Q_{z_i}(b) = \rho\, Q_n(b) + (1-\rho)\, Q_{n+1}(b) = 0,$$

(31.7)
$$\frac{Q_n(b)}{Q_{n+1}(b)} = 1 - \frac{1}{\rho}.$$

The function of b on the left is increasing [13, (3.3.9)] so that ρ is increasing, hence $z_i(b)$ a decreasing function of b.

Thus we can assume that all $z_i(b)$ are non-integers. The same can be assumed about $z_i(c)$; an exception is the case $c = a_{r+1}$ when we aim at the proof of $z_i(b) < z_{i+1}(c)$. Let us define the integers $n_i(b)$ and $n_i(c)$ by

(31.8) $\quad n_i(b) < z_i(b) < n_i(b) + 1, \quad n_i(c) < z_i(c) < n_i(c) + 1$

where

$$n_i(b) + 1 \leqq n_{i+1}(b), \quad n_i(c) + 1 \leqq n_{i+1}(c).$$

We denote by $v(n; x)$ the number of sign changes in (31.5), writing x instead of a, $Q_n(x) \neq 0$, so that $v(n; x)$ is a non-decreasing function in both variables n and x. We have

$$v[n_i(b); b] = v[n_i(c); c] = i-1, \quad v[n_i(b)+1; b] = v[n_i(c)+1; b] = i,$$
$$v(n; b) \leqq v(n; c) \leqq v(n; b) + 1;$$

the last inequality follows from Theorem 19 (a). Now

(31.9)
$$\begin{aligned} &i-1 \leqq v[n_i(b); c] \leqq i, \quad \text{hence} \\ &v[n_i(c); c] \leqq v[n_i(b); c] \leqq v[n_{i+1}(c); c] \end{aligned}$$

so that

$$n_i(c) \leqq n_i(b) \leqq n_{i+1}(c).$$

If $n_i(c) < n_i(b)$ we easily conclude that $z_i(c) < z_i(b)$. Let

$$p = n_i(b) = n_i(c) \;,\; p < z_i(b) < p+1 \;,\; p < z_i(c) < p+1 \;;$$

thus

$$\operatorname{sgn} Q_p(b) = \operatorname{sgn} Q_p(c) = (-1)^{i-1} \;;\; \operatorname{sgn} Q_{p+1}(b) = \operatorname{sgn} Q_{p+1}(c) = (-1)^i.$$

But [14]

$$\begin{vmatrix} Q_p(b) & Q_{p+1}(b) \\ Q_p(c) & Q_{p+1}(c) \end{vmatrix} < 0, \quad \frac{Q_p(b)}{Q_{p+1}(b)} = 1 - \frac{1}{\rho} < \frac{Q_p(c)}{Q_{p+1}(c)} = 1 - \frac{1}{\rho'}, \; \rho < \rho'.$$

Now

$$z_i(b) = \rho p + (1-\rho)(p+1),$$
$$z_i(c) = \rho' p + (1-\rho')(p+1),$$

so that indeed $z_i(c) < z_i(b)$ follows.

Discussing the other inequality $z_i(b) < z_{i+1}(c)$ we set

(31.10) $\quad n_i(b) < z_i(b) < n_i(b) + 1 \;,\; n_{i+1}(c) < z_{i+1}(c) \leqq n_{i+1}(c) + 1$

so that (31.9) holds without change. Again the case $n_i(b) < n_{i+1}(c)$ is trivial so that we may write $p = n_i(b) = n_{i+1}(c)$. The case where $z_{i+1}(c)$ is $= p+1$ is also trivial, so we assume that both $z_i(b)$ and $z_{i+1}(c)$ are non-integers. Now

$$\operatorname{sgn} Q_p(b) = -\operatorname{sgn} Q_p(c) = (-1)^{i-1} \;;\; \operatorname{sgn} Q_{p+1}(b) = -\operatorname{sgn} Q_{p+1}(c) = (-1)^i$$

so that the inequality in question follows as above.

5. Finally we prove

Theorem 21: *Let us use the previous notation and let λ_i be real, $\lambda_k \neq 0$, $\lambda_l \neq 0$. The function*

$$\Phi(z) = \sum_{i=k}^{l} \lambda_i R_{a_i}(z)$$

---

14. If this determinant keeps a constant sign for all b, c such that

$$a_r \leqq b < c \leqq a_{r+1},$$

it must be negative, by Theorem 3. Assume now that it vanishes; two constants A, B would exist not both zero such that $A\,Q_p(x) + B\,Q_{p+1}(x)$ would vanish for $x = b$ and $x = c$. If $B = 0$ this would contradict Theorem 19 (a) so that $A \neq 0$, $B \neq 0$. Now the same reasoning we used for the proof of the Theorem quoted leads to a contradiction.

possesses at most l distinct zeros and at least k nodal zeros for $z \geq 0$.

Remark. Some clarification of the nature of zeros of $\Phi(z)$ is necessary. If $\Phi(z)$ vanishes on an entire segment $[n, n+1]$, then both n and $n+1$ are considered distinct zeros. In the case where $\Phi(z)$ possesses a nodal interval, then corresponding to this interval the left end point is identified as the location of the sign change of $\Phi(z)$.

The proof of Theorem 21 relies heavily on the inequality

$$(31.11) \qquad (-1)^{\frac{k(k+1)}{2}} Q\begin{pmatrix} a_0, a_1, \ldots, a_k \\ n_0, n_1, \ldots, n_k \end{pmatrix} > 0, \qquad k = 0, 1, 2, \ldots$$

with n_i arbitrary satisfying $0 \leq n_0 < n_1 < n_2 < \ldots < n_k$. The proof of (31.11) is given in [8]. It follows immediately from (31.11) that

$$(31.12) \qquad (-1)^{\frac{k(k+1)}{2}} R\begin{pmatrix} a_0, a_1, \ldots, a_k \\ z_0, z_1, \ldots, z_k \end{pmatrix} > 0 \qquad \begin{array}{l} k = 0, 1, 2, \ldots; \\ 0 \leq z_0 < z_1 < \ldots < z_k \end{array}$$

provided that no three successive z_i are contained in a common segment $[n, n+1]$ for some integer $n \geq 0$.

Proof of Theorem 21.

If $\Phi(z)$ possesses in excess of l distinct zeros we contradict relation (31.12).

We turn to the proof of the statement that $\Phi(z)$ possesses at least k nodal zeros. Suppose to the contrary. Let $z_1 < z_2 < \ldots < z_t$, $t < k$, denote the totality of nodal zeros of $\Phi(z)$. By virtue of the convention of the Remark, the z_i are necessarily located in distinct segments

$$n_i \leq z_i < n_{i+1}.$$

Moreover if $z_i = n_i$ then $z_{i-1} < n_i - 1$.

We now determine certain real constants μ_r not all zero with the property that

$$(31.13) \qquad \psi(z) = \sum_{r=0}^{t} \mu_r R_{a_r}(z), \quad \psi(z_i) = 0, \qquad i = 1, 2, \ldots, t.$$

Explicitly, let

$$(31.14) \qquad \psi(z) = \begin{vmatrix} R_{a_0}(z_1) & R_{a_1}(z_1) & \dots & R_{a_t}(z_1) \\ R_{a_0}(z_2) & R_{a_1}(z_2) & \dots & R_{a_t}(z_2) \\ \vdots & \vdots & \dots & \vdots \\ R_{a_0}(z_t) & R_{a_1}(z_t) & \dots & R_{a_t}(z_t) \\ R_{a_0}(z) & R_{a_1}(z) & \dots & R_{a_t}(z) \end{vmatrix}.$$

By virtue of (31.12) the coefficient μ_t of $R_{a_t}(z)$ in $\psi(z)$ is non-zero of actual sign $(-1)^{t(t-1)/2}$. We now show that the values z_i are the only nodal zeros of $\psi(z)$. In fact, consider $z < z_1$; we may interchange the rows of (31.14) and compare with (31.12). It follows that on this interval $(-1)^{t(t-1)/2}\,\psi(z) \geqq 0$ and $(-1)^{t(t-1)/2}\,\psi(0) > 0$. Let now $z_1 < z < z_2$; we interchange the rows arranging them according to the inequalities

$$z_1 < z < z_2 < \dots < z_t\,.$$

Again appealing to (31.12) we infer that $(-1)^{[t(t-1)/2]+1}\,\psi(z) > 0$, and so on.

Thus, we see that the function $\psi(z)$ is non identically zero and only changes sign at z_i. We conclude that $\psi(n) \cdot \Phi(n)$ is of one sign independent of n and is not everywhere zero. But $\sum_{n=0}^{\infty} \pi_n\, \psi(n)\, \Phi(n)$ exists [recall that $\sum_{n=0}^{\infty} \pi_n\, [Q_n(a_i)]^2 < \infty$ for every a_i] and is therefore non-zero. On the other hand, since

$$\sum_{n=0}^{\infty} \pi_n\, Q_n(a_i)\, Q_n(a_j) = 0$$

when $a_i \neq a_j$ we deduce that $\Sigma\, \pi_n\, \psi(n)\, \Phi(n) = 0$ which is a contradiction. The proof of Theorem 21 is hereby complete.

6. We close this section with a brief discussion regarding the oscillation properties of $R_a(z)$ when the spectrum of ψ is not necessarily discrete.

One result in this direction is the following.

Theorem 22: Let $0 < b < c$ and define $n_1(b), n_2(b), \dots,$ and $n_1(c), n_2(c), \dots,$ representing the indices where the respective sign changes of $R_b(n)$ and $R_c(n)$ occur [see (31.8)]. Then between two successive $n_k(b)$ and $n_{k+1}(b)$ there exists

at least one sign change of $R_c(n)$, i.e. $n_k(b) \leq n_l(c) \leq n_{k+1}(b)$ for some l.

Proof. By the definition of $n_k(b)$, $Q_{n_k+1}(\xi)$ is the first polynomial in the sequence (Q_n) which possesses k zeros in the interval $[0, b]$. Suppose $Q_{n_k+1}(\xi)$ has r zeros located in $(b, c]$. We examine $Q_{n_{k+1}+1}(\xi)$ and observe that this polynomial possesses $k+1$ zeros prior to or equal to b and at least $r' \geqq r-1$ zeros in the interval $(b, c]$. (The interlacing character of the roots of successive polynomials is needed here.)

We now consider two cases.

(i) $r' > r-1$. The polynomial $Q_{n_{k+1}}(\xi)$ has $k+r'$ zeros in $(0, c]$ and $Q_{n_{k+1}+1}(\xi)$ has $k+r+(r'-r+1)$ zeros in $(0, c]$. Amongst the sequence $Q_{n_k+1}, Q_{n_k+2}, \ldots, Q_{n_{k+1}+1}$ there exists a first polynomial Q_{n_k+t} which has $k+r+1$ roots in $(0, c]$. Hence $n_{k+r}(c) = n_k + t - 1$ which obviously is included in the interval of $n_k(b)$ to $n_{k+1}(b)$.

(ii) $r' = r-1$. We now examine $Q_{n_k}(\xi)$ which possesses $k-1$ roots in $[0, b)$ and r or $r+1$ roots in $(b, c]$. We first show that the second possibility does not happen. Assuming this, $Q_{n_{k+1}+1}$ has $r-1$ roots in $(b, c]$ which means there exists a pair of roots for Q_{n_k} located in $(b, c]$ such that no root of $Q_{n_{k+1}+1}$ separates them. This violates a classical result on orthogonal polynomials [13, Theorem 3.3.3]. Hence $Q_{n_k}(\xi)$ possesses only r roots in $(b, c]$. But then $n_{k+r-1}(c) = n_k(b)$ and the proof of the theorem is complete.

§ 32. Determinants of the Turân type outside the spectrum.

In this section we discuss the sign of the determinants occuring in Theorem 5 for values of x outside the spectrum; we consider in particular the ultraspherical and Laguerre polynomials. The results are however valid more generally and not dependent on the order of the determinant being even or odd. In dealing with these questions the basic tool will be the following determinantal identity. Let X, Y and Z be linear sets on the real line representing the ranges of the variables x, y, z. If

$$M(x, y) = \int K(x, t)\, L(t, y)\, d\sigma(t) \tag{32.1}$$

where σ is a sigma-finite regular measure and the integral (32.1) is assumed to converge absolutely, then

$$(32.2)\quad M\begin{pmatrix} x_1, x_2, \ldots, x_l \\ y_1, y_2, \ldots, y_l \end{pmatrix}$$
$$= \int_{t_1<t_2<\ldots<t_l} \cdots \int K\begin{pmatrix} x_1, x_2, \ldots, x_l \\ t_1, t_2, \ldots, t_l \end{pmatrix} L\begin{pmatrix} t_1, t_2, \ldots, t_l \\ y_1, y_2, \ldots, y_l \end{pmatrix} d\sigma(t_1)\, d\sigma(t_2) \ldots d\sigma(t_l).$$

The symbol $M\begin{pmatrix} x_1, x_2, \ldots, x_l \\ y_1, y_2, \ldots, y_l \end{pmatrix}$ stands for the determinant $[M(x_\mu, y_\nu)]$ where $\mu, \nu = 1, 2, \ldots, l$. The other factors appearing in the integrand of (32.2) have a similar meaning. The proof may be found in [11, Vol. 1, p. 48, Problem 68]. The main use of (32.2) for our purposes will be the absolutely continuous case, i.e. when $d\sigma(t) = \omega(t)\, dt$, and the case where $d\sigma(t)$ is the counting measure of the non-negative integers. In the latter circumstance the integrals of (32.2) are replaced by finite or infinite sums.

1. Our first application of (32.2) is on moment sequences. Let

$$(32.3)\qquad a_n = \int_0^\infty [u(t)]^n\, \omega(t)\, dt$$

where $u(t)$ is positive and strictly monotone increasing for $t > 0$, moreover $\omega(t) \geqq 0$ for $t > 0$ and positive on some interval.

Lemma 8: *Let l be a positive integer. For any two collections of integers arranged so that*

$$0 \leqq m_1 < m_2 < \ldots < m_l\,; \quad 0 \leqq n_1 < n_2 < \ldots < n_l,$$

we have the determinant inequality $[a_{m_\mu + n_\nu}]_1^l > 0$.

Proof. We use the representation

$$(32.4)\qquad a_{m+n} = \int_0^\infty [u(t)]^m\, [u(t)]^n\, \omega(t)\, dt.$$

Since the determinant $[u(t_\mu)]^{n_\nu}$ of the generalized Vandermonde character is positive whenever $0 < t_1 < t_2 < \ldots < t_l$ and $0 \leqq n_1 < n_2 < \ldots < n_l$ [cf. 11, Vol. 2, p. 45, Problem 48], we deduce the desired result by applying (32.2) to (32.4).

Consider, for instance, the ultraspherical polynomials

$$P_n^{(\lambda)}(x) \,/\, P_n^{(\lambda)}(1) = Q_n(x) \qquad (\lambda > 0)$$

with $x > 1$. The representation (5.8), i.e.

$$(32.5) \qquad Q_n(x) = \pi^{-\frac{1}{2}} \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)} \int_0^\pi (x + \sqrt{x^2-1}\cos\varphi)^n \sin^{2\lambda-1}\varphi \, d\varphi, \ \lambda>0,$$

is of the form (32.3). Referring to the Lemma we obtain especially that

$$(32.6) \qquad (\operatorname{sgn} x)^{nl} \cdot \left[\frac{P^{(\lambda)}_{n+\mu+\nu}(x)}{P^{(\lambda)}_{n+\mu+\nu}(1)} \right]_0^{l-1} > 0, \quad x > 1 \text{ or } x < -1 \ ; \ \lambda > 0.$$

Another application of the Lemma refers to the Laguerre polynomials $R_n(x) = n!\ L_n^{(\alpha)}(-x)$. (The reader may observe that the present normalization is not the usual one.) In view of the representation [13, (5.4.1)]

$$(32.7) \qquad n!\, L_n^{\alpha}(-x) = e^{-x} x^{-\frac{\alpha}{2}} \int_0^\infty e^{-t} t^{\frac{\alpha}{2}} \cdot t^n I_\alpha(2\sqrt{tx})\, dt$$

we infer, as a special circumstance of Lemma 8, that

$$[R_{n+\mu+\nu}(x)]_0^{l-1} > 0 \quad \text{for} \quad x > 0.$$

2. We now proceed with the analysis of the determinant of the Turán type for

$$Q_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) \,/\, L_n^{(\alpha)}(0)$$

where $x < 0$. We claim that

$$(32.8) \qquad (-1)^{l(l-1)/2} \, [Q^{(\alpha)}_{n+\mu+\nu}(x)]_0^{l-1} > 0 \quad \text{for} \quad x < 0;$$

more generally, if m_μ and n_ν are subject to the conditions of the Lemma:

$$(32.9) \qquad (-1)^{l(l-1)/2} \, [Q^{(\alpha)}_{n+m_\mu+n_\nu}(x)]_1^l > 0 \quad \text{for} \quad x < 0.$$

As to (32.8) cf. the closing remarks of § 17. We shall prove the more general assertion (32.9).

To this end, consider $x < 0$ fixed and write in accordance with (16.9) and (16.10)

$$(32.10) \qquad c_n = \frac{Q_n^{(\alpha)}(x)}{\Gamma(\alpha+1)} = \sum_{\rho=0}^{\infty} \binom{n}{\rho} \frac{(-x)^\rho}{\Gamma(\alpha+\rho+1)} = \sum_{\rho=0}^{\infty} \binom{n}{\rho} d_\rho ,$$

$$d_\rho = \frac{(-x)^\rho}{\Gamma(\alpha+\rho+1)} .$$

Using

$$\binom{m+n}{\rho} = \sum_{l=0}^{\infty} \binom{m}{l} \binom{n}{\rho-l}$$

we secure the expression

$$(32.11) \qquad c_{m+n} = \sum_{\rho=0}^{\infty} d_\rho \sum_{l=0}^{\infty} \binom{m}{l} \binom{n}{\rho-l} = \sum_{l=0}^{\infty} \binom{m}{l} \sum_{\rho=0}^{\infty} \binom{n}{\rho} d_{\rho+l}$$

the last resulting by an interchange of the summation and then performing a suitable change of variable. It is now convenient to introduce the kernel functions

$$K(m, n) = \binom{m}{n}, \quad L(m, n) = d_{m+n}, \quad m, n = 0, 1, 2, \ldots .$$

For our immediate purposes we record the following facts concerning these kernels.

(i) A direct calculation or alternately applying (32.2) to the representation formula

$$(32.12) \quad d_{m+n} = \frac{(-x)^m (-x)^n}{\Gamma(\alpha+m+n+1)} = \frac{(-x)^m (-x)^n}{\Gamma(m+\frac{\alpha}{2})\Gamma(n+1+\frac{\alpha}{2})} \int_0^1 t^{m+\frac{\alpha}{2}-1} (1-t)^{n+\frac{\alpha}{2}} dt$$

implies that for $0 \leqq m_1 < m_2 < \ldots < m_l$; $0 \leqq n_1 < n_2 < \ldots < n_l$ where l is a positive integer but otherwise arbitrary,

$$(32.13) \qquad (-1)^{l(l-1)/2} [L(m_\mu, n_\nu)]_1^l = (-1)^{l(l-1)/2} [d_{m_\mu+n_\nu}]_1^l > 0 .$$

The argument of (32.12) works only for $\alpha > 0$; the result is actually valid for all $\alpha > -1$. We omit the formal calculation.

(ii) We verify by induction that

$$(32.14) \qquad [K(m_\mu, n_\nu)]_1^l = \left[\binom{m_\mu}{n_\nu}\right]_1^l \geqq 0$$

for the same values of m_μ, n_ν as above; l is again positive. (It is easily ascertained that this determinant is not identically zero.)

Indeed, $n > n'$,

$$\binom{n}{k} - \binom{n'}{k} = \binom{n'}{k-1} + \binom{n'+1}{k-1} + \dots + \binom{n-1}{k-1}$$

so that subtracting the $\mu - 1^{\text{th}}$ row from the μ^{th} row:

$$(32.15) \qquad \binom{m_\mu}{n_\nu} - \binom{m_{\mu-1}}{n_\nu} = \sum_{l=m_{\mu-1}}^{m_\mu-1} \binom{l}{n_\nu-1}.$$

This holds also for the first row, $\mu = 1$, $m_0 = 0$. Decomposing the rows of this determinant into sums, the resulting determinants will be of the same type as the original one where n_ν is replaced by $n_\nu - 1$. We note that the quantity l appearing in (32.15) exceeds all l occuring in the previous difference (μ replaced by $\mu - 1$). Also, if $n_1 = 0$, the determinant can be reduced to one of order $l - 1$.

Thus we have the possibility of induction, reducing the given determinant step by step either to one of order $l - 1$ or to one of order l in which $n_1, n_2, \dots, n_l$ coincide with the integers $0, 1, \dots, l - 1$. This is of course a determinant of the Vandermonde type.

Now we define

$$M(m, n) = \sum_{\rho=0}^{\infty} \binom{m}{\rho} d_{\rho+n} = \sum_{\rho=0}^{\infty} K(m, \rho) L(\rho, n).$$

Applying (32.2) we obtain that

$$(32.14) \qquad (-1)^{l(l-1)/2} [M(m_\mu, n_\nu)]_1^l > 0 \quad \text{for} \quad \begin{array}{l} 0 \leqq m_1 < m_2 < \dots < m_l, \\ 0 \leqq n_1 < n_2 < \dots < n_l. \end{array}$$

Since (32.11) can be written as

$$(32.15) \qquad c_{m+n} = \sum_{l=0}^{\infty} K(m, l) M(l, n),$$

the assertion (32 9) follows.

The method just completed is capable of several generalizations to which we will return on another occasion.

3. The Turán determinant for the classical discrete polynomials for values outside the smallest convex set enclosing the spectrum can be discussed by these same methods. As typical illustrations we record the conclusions without proof for the case of the Poisson-Charlier and the Meixner polynomials for some special choices of the parameters.

(a) Consider the Poisson-Charlier polynomials $\gamma_n = c_n(a\,; x)$, see (5.18). For $x < 0$ we have

$$[\gamma_{m_\mu+n_\nu}]_1^l > 0\,; \qquad \begin{array}{l} 0 \leqq m_1 < m_2 < \dots < m_l\,, \\ 0 \leqq n_1 < n_2 < \dots < n_l\,. \end{array} \tag{32.16}$$

(b) Consider the Meixner polynomials $\lambda_n = M_n(1\,, \gamma\,; x)$, see (5.22);

$$\lambda_n = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k}\left(1-\frac{1}{\gamma}\right)^k = \sum_{k=0}^{n}\binom{n}{k}\binom{-x+k-1}{k}\left(\frac{1}{\gamma}-1\right)^k. \tag{32.17}$$

If x is a negative integer, we have

$$(-1)^{l(l-1)/2}\,[\lambda_{m_\mu+n_\nu}]_1^l > 0 \qquad \begin{array}{l} 0 \leqq m_1 < m_2 < \dots < m_l\,, \\ 0 \leqq n_1 < n_2 < \dots < n\,. \end{array} \tag{32.18}$$

When x is not an integer the sign of (32.18) is sensitive to the value of l and the specific location of x. At this point we shall not enter into a discussion of the various possibilities.

§ 33. Open problems.

It seems that the present investigation is only the beginning of a trend which is rich in open problems and promising topics. In the following lines we point out a few such problems. We do not possess complete solutions in any case so that the assertions formulated below have the character of conjectures. Some of these problems were already mentioned in the course of our discussions. Others appear here for the first time. In most of the problems we append some possibly useful remarks.

1. Jacobi polynomials.

Let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomials,

$$Q_n(x) = P_n^{(\alpha,\beta)}(x) \,/\, P_n^{(\alpha,\beta)}(1)\,,$$

cf. § 3.3. If $\alpha \leqq \beta$, we raise the question whether or not the Turán inequality holds, i.e., is it true that

$$\begin{vmatrix} Q_{n-1}(x) & Q_n(x) \\ Q_n(x) & Q_{n+1}(x) \end{vmatrix} < 0, \qquad -1 < x < 1 ? \tag{33.1}$$

The restriction $\alpha \leqq \beta$ is essential in view of § 3.3.

2. A result of Forsythe [4] states that, $Q_n(x) = P_n(x)$,

$$\begin{vmatrix} Q_{2n-1}(x) & Q_{2n+1}(x) \\ Q_{2n+1}(x) & Q_{2n+3}(x) \end{vmatrix} < 0, \qquad -1 < x < 1 \ , \ x \neq 0 ,$$

in the case of Legendre's polynomials; this fact is derivable from (33.1) by the specialization $\alpha = 0$, $\beta = \frac{1}{2}$, on account of the relationship

$$P_{2n+1}(x) = x\, P_n^{(0\,,\,1/2)}(2x^2 - 1)\,.$$

Another very simple instance is

$$\alpha = -\tfrac{1}{2}, \quad \beta = \tfrac{1}{2}$$

[13, (4.1.8)].

3. “Turán's inequality (in the 2×2 case) is not valid if x is outside of the spectrum”.

The following three examples of this assertion have not been examined in the present paper. Their elucidations would be of interest:

(a) Krawtchouk polynomials,

(b) Meixner polynomials,

(c) Tchebychev's polynomials of a discrete measure.

[Problem (a) seems rather easy: we have (20.4) for arbitrary x and we can apply Darboux's method, cf. § 30.5 (b); as to (b) use (20.2) where we can exploit the transformation of the hypergeometric function ${}_1F_1$ [2, Vol. 1, p. 108, (1)]; case (c) seems difficult, and we offer no real insights. It might be advantageous to write $x = m + 1/2$, and choose m, n, N as large integers appropriately related to each other.]

4. Hankel forms.

Let $P_n(x)$ be Legendre's polynomial, $D = \dfrac{d}{dx}$. We conjecture that

the determinant

$$[D^{\mu+\nu} P_n(x)]_0^{l-1}\ ,\ \mu\,,\nu=0,\,1,\,2,\,\ldots,\,l-1\,;\quad n\geqq l-1\,,$$

as a function of x, keeps a constant sign for all real x if l is even; it keeps a constant sign for $x>1$, and for $x<-1$ if l is odd. A similar statement should apply to the general ultraspherical polynomial.

[In this connection we refer the reader to § 30.1 where the corresponding assertion involving the Laguerre polynomials is discussed.]

5. Consider the function

$$\varphi(x)=\begin{vmatrix} Q_n(x) & Q_{n+h}(x) \\ Q_{n+k}(x) & Q_{n+h+k}(x) \end{vmatrix} \tag{33.2}$$

where $Q_n(x)$ represents either the ultraspherical $(\lambda>0)$ or Laguerre or Hermite polynomials normalized as in Theorem 5. Here n, h and k are fixed integers, $h>0$, $k>0$ but otherwise arbitrary. It is easy to prove that $\varphi(x)$ never vanishes outside the spectrum.

We conjecture that (33.2) has precisely $h-1+k-1$ zeros, counting multiplicities, located interior to the spectrum.

There is substantial evidence to support this. For example if either h or $k=1$, the assertion is correct by virtue of Theorem 10. If $n=0$ and $h=2$ or 3, it is quite easy to establish the conjecture in the cases of the ultraspherical and Hermite polynomials.

We indicate the argument briefly in the case of the Hermite polynomials, $Q_n(x)=H_n(x)$ [13, 5.5]. We observe that for $n=0$, $h=2$ the determinant (33.2) reduces to

$$\varphi(x)=H_{k+2}(x)-H_2(x)\,H_k(x)\,,$$

and in view of the orthogonality property this has at least $k-2$ nodal zeros. Since $\varphi(x)$ is even or odd as k is even or odd, these zeros are symmetrically located around $x=0$. Also, $\varphi(x)<0$ for $|x|$ large. Substituting $x=0$ in (33.2) we find in view of [13, (5.5.5)]

$$H_{2m}(0)=(-1)^m\frac{(2m)!}{m!}\quad\text{and}\quad H'_{2m+1}(0)=(-1)^m\frac{(2m+2)!}{(m+1)!}$$

that $\operatorname{sgn}\varphi(0)=(-1)^{m+1}$ for $k=2m$ while $\varphi(0)=0$ and $\operatorname{sgn}\varphi'(0)=(-1)^{m+1}$

for $k = 2m + 1$. Combining these facts suitably we obtain the desired conclusion. The ultraspherical case is treated by similar methods.

A slight refinement of this argument establishes the conjecture also in the case where $n = 0$ and $h = 3$.

Another case of this conjecture involving the specific form

$$\varphi(x) = \begin{vmatrix} Q_{2n-1}(x) & Q_{2n+1}(x) \\ Q_{2n+1}(x) & Q_{2n+3}(x) \end{vmatrix}$$

for the ultraspherical polynomials, with parameter in the range $\frac{1}{2} \leq \lambda \leq 1$, has been studied by A. E. Danese.[15] He shows that $\varphi(x)$ has a double zero at $x = 0$ and no other zero interior to the interval $[-1, 1]$.

6. In order to state our next problem, we formulate a stronger version of the Sturm properties: Let $\{Q_n(x);\ n = 0, 1, 2, \ldots\}$ denote a system of functions. We say that $\{Q_n(x)\}$ constitute a strong Sturm sequence (S.S.S) provided the following two properties are satisfied.

(a) Let $\varphi(x) = \sum_{r=k}^{l} \lambda_r Q_r(x)$ for λ_r real and not all zero.

Then $\varphi(x)$ has at least k nodal zeros and at most l distinct zeros.

(b) Between two successive zeros of $Q_n(x)$ there exists at least one zero of $Q_m(x)$ for each $m > n$.

It is a familiar fact that every set of orthogonal polynomials constitutes a S.S.S. [see 13, Theorem 3.3.3].

In this connection we refer to Theorem 16 (§ 30.1) and Theorem 21 (§ 31.5) which exhibit Sturm sequences possessing property (a). We now raise the general question inquiring under what circumstances any of the determinantal polynomial Sturm systems introduced in this paper actually are S.S.S. More specifically, when does either property (a) or (b) or both hold for the systems occuring in Theorems 2, 6, 9, 10, 12 or 14.

7. On the basis of our preceding discussions we may conjecture several new characterizations of the classical orthogonal polynomials.

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15. A. E. Danese, Some identities and inequalities involving ultraspherical polynomials, Duke Mathematical Journal, Vol. 26, 1959, pp. 349—360.

(a) It was noted in § 24 that each of the classical polynomials satisfies a relation of the form

$$r(x) P'_n(x) = \mu_n [P_n(x) + c(x) P_{n+1}(x)] \rho(x), \qquad n = 0, 1, 2, \ldots,$$

where $c(x)$ is a polynomial in x. Does this type of relation characterize the classical orthogonal polynomials perhaps subject to some appropriate smoothness conditions on $r(x)$ and $\rho(x)$?

(b) Each of the classical orthogonal polynomials has the property that $\{Q'_n\}$ constitutes a Sturm set. We inquire whether this property coupled with the fact that $\{Q_n\}$ is orthogonal implies that this system is one of the classical types?

(c) A known criterion [cf. 13, p. 106] states that if $\{Q_n\}$ and $\{Q'_n\}$ each constitute a family of orthogonal polynomials, then Q_n is of the classical type. Suppose $\{Q_n\}$ and $\}Q_n^{(r)}\}$ for some fixed $r > 1$ are orthogonal systems of polynomials. Is it still true that $\{Q_n\}$ is a classical system?

8. We have not been able to prove the higher order Turán inequality in the case of Tchebychev's polynomials of a discrete measure [cf. § 5 (d), § 20 (d), and the generalization in § 21]. These polynomials correspond to a discrete version of the Legendre and ultraspherical polynomials. Of course, the inequality to be proved refers only to the spectrum.

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ON THE METHOD OF STEEPEST DESCENT

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The method of steepest descent involves integrals of the form

$$J = \int_W g(z)\, e^{f(z)}\, dz\,,$$

where $f(z)$ and $g(z)$ are analytic functions of z on the integration path W. This path contains a saddle point ζ of the function $f(z)$, namely a point where $f'(z)$ assumes the value zero. That W is a curve of steepest descent means here that $f(z_1) - f(z_2) \geqq 0$ for any two points z_1 and z_2 on W with the property that z_1 lies on the arc $(\zeta\,, z_2)$ of W.

Under general conditions it is possible to find on W on both sides of the saddle point ζ two points α and β such that the integral J is, with a high degree of accuracy, approximately equal to the contribution of the arc $(\alpha\,, \beta)$. In simple cases it is easy to find an upper bound for the absolute value of the contribution of the part of W outside this arc, but if the functions $f(z)$ and $g(z)$ depend, not only on z, but also on a number of variable parameters, then in general we know practically nothing about the form of the curve of steepest descent with the result that it is not possible to find the required upper bound. In view of this fact I consider in this paper integrals of the form

$$j = \int_\Lambda g(z)\, e^{f(z)}\, dz\,, \tag{1}$$

where the path is a curve of steepest descent for the function $e^{f(z)}$. This means here that $f(z_1) - f(z_2)$ is positive for any two points z_1 and z_2 with the property that z_1 lies on Λ before z_2. We assume that the initial point η_0 of the integration path is finite; the endpoint of the path may lie at infinity.

Under general conditions I shall deduce an upper bound for the absolute value of the integral j, even if we know practically nothing about the form of the integration path.

It is even not necessary that the integration path is a curve of steepest descent for the function $e^{f(z)}$. It is sufficient that the path Λ is a continuous rectifiable curve with the property that any two points z_1 and z_2 on Λ such that z_1 lies on Λ before z_2 satisfy the inequality

$$-\lambda \leqq \arg(f(z_1) - f(z_2)) \leqq \lambda, \tag{2}$$

where λ denotes a given number $\geqq 0$ and $< \frac{\pi}{2}$ independent of z_1 and z_2.

If $\lambda = 0$, then Λ is a curve of steepest descent for the function $e^{f(z)}$. I call a continuous rectifiable curve Λ with property (2) a curve of descent with angle λ for the function $e^{f(z)}$. It is clear that $|e^{f(z)}|$ decreases as z traverses such a curve.

To give some idea about the theorems to which this paper is devoted I deduce first a simple result.

Theorem 1: *If Λ is a curve of descent for the function $e^{f(z)}$ with angle $\lambda \left(0 \leqq \lambda < \frac{\pi}{2}\right)$ and initial point η_0 and if the functions $f(z), g(z) \neq 0$ and $\tau(z) > 0$ are differentiable along Λ with*

$$|f'(z)| \geqq \tau(z) \quad \text{and} \quad \left|\frac{g'(z)}{g(z)}\right| + \left|\frac{\tau'(z)}{\tau(z)}\right| \leqq \frac{1}{2}|f'(z)| \cos\lambda, \tag{3}$$

then

$$\left|\int_\Lambda g(z) e^{f(z)}\, dz\right| < \frac{2e}{\tau(\eta_0)\cos\lambda} |g(\eta_0) e^{f(\eta_0)}|. \tag{4}$$

Remark: Roughly Theorem 1 can be formulated as follows: if at each point z of Λ the first derivative $f'(z)$ is in absolute value large enough and $g(z)$ does not change too rapidly, then the theorem yields an upper bound for the absolute value of the integral under consideration, since in this case it is possible to find a positive function $\tau(z)$ such that the inequalities (3) hold.

The factor $\frac{1}{2}$ occurring on the right hand side of the second inequality (3) may not be replaced by 1, for if we choose $\lambda = 0$, $g(z) = e^{-f(z)}$ and $\tau(z) = \tau$, where τ denotes a positive number independent of z such that $|f'(z)| \geqq \tau$, then the inequalities (3) hold with the factor $\frac{1}{2}$ replaced by 1, but the assertion does not hold, since the integrand $g(z)e^{f(z)}$ is equal to 1, so

that the absolute value of the integral under consideration is equal to the distance between the initial and the endpoint of Λ and may be therefore infinite.

Proof: For each point z on Λ we call the successor of z the point w which lies on Λ behind z with

$$\int_z^w |f'(t)|\,|dt| = 1, \tag{5}$$

if such a point w exists; otherwise I call the endpoint of Λ the successor of z.

If z is a point on Λ with successor w and if t_1 and t_2 denote arbitrary points lying on the closed arc (z, w), then

$$\log\left|\frac{g(t_1)}{g(t_2)}\right| \leqq \left|\log\frac{g(t_1)}{g(t_2)}\right| = \left|\int_{t_2}^{t_1} \frac{g'(t)}{g(t)}\,dt\right| \leqq \int_{t_2}^{t_1} \left|\frac{g'(t)}{g(t)}\right|\,|dt|.$$

Replacing in this formula g by τ we find by means of the second inequality (3) for any four points t_1, t_2, t_3, t_4 lying on the closed arc (z, w)

$$\log\left|\frac{g(t_1)}{g(t_2)}\right| + \left|\log\frac{\tau(t_3)}{\tau(t_4)}\right| \leqq \frac{1}{2}(\cos\lambda)\int_z^w |f'(t)|\,|dt| \leqq \frac{1}{2}\cos\lambda$$

by (5), hence

$$\left|\frac{g(t_1)}{g(t_2)}\right| \frac{\tau(t_3)}{\tau(t_4)} \leqq e^{1/2\cos\lambda} \leqq e^{1/2}. \tag{6}$$

If z is a point on Λ whose successor w does not coincide with the endpoint of Λ, then (5) holds. Since Λ is a curve of descent with angle λ for the function $e^{f(z)}$ we have therefore

$$\operatorname{Re}(f(w) - f(z)) = \operatorname{Re}\int_z^w f'(t)\,dt \leqq -(\cos\lambda)\int_z^w |f'(t)|\,|dt| = -\cos\lambda.$$

Consequently

$$e^{f(w)}| \leqq e^{-\cos\lambda}\, e^{f(z)}, \tag{7}$$

so that according to (6),

$$\left|\frac{g(w)}{\tau(w)}\,e^{f(w)}\right| \leqq e^{-1/2\cos\lambda}\cdot\left|\frac{g(z)}{\tau(z)}\,e^{f(z)}\right|. \tag{8}$$

If z is a point on Λ whose successor w may coincide with the end-point of Λ, then

$$\int_z^w \tau(t)\,|dt| \leqq \int_z^w |f'(t)|\,|dt| \leqq 1 \text{ and } |e^{f(t)}| \leqq |e^{f(z)}|$$

for each point t on the arc (z, w), so that according to (6)

$$(9)\qquad \begin{cases} \left|\int_z^w g(t)\,e^{f(t)}\,dt\right| \leqq e^{1/2}\left|\dfrac{g(z)}{\tau(z)}\,e^{f(z)}\right| \int_z^w \tau(t)\,|dt| \\ \qquad\qquad \leqq e^{1/2}\left|\dfrac{g(z)}{\tau(z)}\,e^{f(z)}\right|. \end{cases}$$

Consider the points $\eta_0, \eta_1, \eta_2, \dots,$ where η_0 is the initial point of the path Λ and where $\eta_{h+1}\,(h \geqq 0)$ is the successor of η_h. If we find in this way only a finite number of points η_h on Λ, then the last point η_{H+1} coincides with the endpoint of Λ. If we find infinitely many points η_h, then I put $H = \infty$; if $h \to \infty$, then it follows from (7) that $e^{f(\eta_h)} \to 0$, so that η_h tends to the endpoint of Λ. We find therefore for finite H and also for $H = \infty$

$$\begin{aligned} \left|\int_\Lambda g(z)\,e^{f(z)}\,dz\right| &\leqq \sum_{h=0}^{H}\left|\int_{\eta_h}^{\eta_{h+1}} g(z)\,e^{f(z)}\,dz\right| \\ &\leqq e^{1/2}\sum_{h=0}^{H}\left|\frac{g(\eta_h)}{\tau(\eta_h)}\,e^{f(\eta_h)}\right| && \text{by (9)} \\ &\leqq e^{1/2}\left|\frac{g(\eta_0)}{\tau(\eta_0)}\,e^{f(\eta_0)}\right|\left|\sum_{h=0}^{\infty} e^{-1/2\,h\cos\lambda}\right| && \text{by (8)} \\ &= e^{1/2+1/2\cos\lambda}\,(e^{1/2\cos\lambda}-1)^{-1}\left|\frac{g(\eta_0)}{\tau(\eta_0)}\,e^{f(\eta_0)}\right|, \end{aligned}$$

which gives the required result (4), since $e^{1/2\cos\lambda} - 1 > \frac{1}{2}\cos\lambda$.

It is true that under certain circumstances Theorem 1 yields an upper bound for the absolute value of the integral under consideration, even if we do not know the form of the integration path. However, to this end we need the inequality $|f'(z)| \geqq \tau(z)$ for each point of the integration path. If the function f and the form of the integration path depend on one or more variable parameters, then it may happen that for some values of

these parameters $|f'(z)|$ is somewhere on the path very small or even equal to zero. These values of the parameters must then be left out of consideration, but the determination of these exceptional values of the parameters is, apart from the very simple cases, long or even impracticable. There is more: if we know these values, then we must try to find an upper bound for the absolute value of the integral for these values of the parameters in which case Theorem 1 can not be applied. In view of this fact it is the purpose of this paper to replace Theorem 1 by deeper results which yield an upper bound for the absolute value of the integral under consideration even if the integration path contains points z where $|f'(z)|$ is very small or even zero.

Theorem 2: For each positive integer n it is possible to find two positive numbers γ and c depending only on n such that the inequality

$$(10) \qquad \left| \int_{\Lambda} g(z)\, e^{f(z)}\, dz \right| < \frac{c}{\tau(\eta_0)\cos\lambda} \left| g(\eta_0)\, e^{f(\eta_0)} \right| ,$$

where the integration path Λ is a curve of descent for $e^{f(z)}$ with angle λ $\left(0 \leq \lambda < \dfrac{\pi}{2}\right)$ and initial point η_0, certainly holds if each point on Λ satisfies the following

Condition: It is possible to find at least one positive integer $l \leqq n$ (l may depend on z) such that $f(w)$ and $g(w)$ are analytic functions of w in the circle

$$(11) \qquad |w - z| < \sigma^{-1}(z) ,$$

where

$$(12) \qquad \sigma(z) = \sum_{h=1}^{l} | f^{(h)}(z) |^{1/h} \geqq \tau(z) > 0 ,$$

with the property that each point w which lies on Λ behind z with (11) satisfies the inequality

$$(13) \qquad \tau(w) \geqq (1 - \gamma\cos\lambda)\, \tau(z)$$

and that each point w with (11) satisfies the inequalities

$$(14) \qquad | f^{(l+1)}(w) | \leqq \sigma^{l+1}(z) \quad \text{and} \quad | g(w) | \leqq (1 + \gamma\cos\lambda) | g(z) | .$$

Remark: Roughly Theorem 2 can be formulated as follows: If it is possible to find for each point z on Λ a positive integer $l \leqq n$ such that

at least one of the l derivatives $f^{(h)}(z)$ $(h = 1, 2, \ldots, l)$ is in absolute value large enough and that in a certain neighborhood of the path $|f^{(l+1)}(w)|$ is small enough and $g(w)$ does not change too rapidly, then Theorem 2 yields an upper bound for the absolute value of the integral under consideration; indeed then we can choose a function $\tau(z)$ which satisfies (12) and which does not change too rapidly so that (13) holds. This is a great improvement in comparison with Theorem 1 which is useless if Λ contains at least one point z where $|f'(z)|$ is very small.

To give some idea about the significance of Theorem 2 I first treat an application in which ω denotes an arbitrary element of a given unbounded point set lying in the complex plane or on a Riemann surface. We introduce a fixed positive integer n and moreover n fixed real distinct numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ each $\neq 0$; "fixed" means: independent of ω. Put

$$f(z) = \sum_{k=1}^{n} u_k z^{\alpha_k}, \tag{15}$$

where $u_1, u_2, \ldots, u_n$ denote real or complex numbers, not all zero, which may depend on ω. The problem is to determine for large $|\omega|$ an upper bound for the absolute value of the integral j mentioned in (1), where Λ is a curve of descent with fixed angle λ $\left(0 \leq \lambda < \frac{\pi}{2}\right)$ for the function $e^{f(z)}$, and where $g(z)$ denotes a function of z which may depend also on ω; also the initial point η_0 of the integration path Λ may depend on ω.

This problem involves so many parameters that we know practically nothing about the form of the integration path, but nevertheless we shall find under a general and simple condition an upper bound for the order of magnitude of the integral.

Theorem 3: Condition (1). Assume that

$$U(z) = \sum_{k=1}^{n} |u_k z^{\alpha_k}| \tag{16}$$

tends for each point z on Λ, uniformly in z, to infinity as $|\omega| \to \infty$.

(2) Assume for each point z on Λ that $g(w)$ denotes for

$$|w - z| < |z| (U(z))^{-1/n} \tag{17}$$

an analytic function of w which satisfies for large $|\omega|$ the order relation

$$g(w) = (1 + o(1))\, g(z) \tag{18}$$

uniformly in z and w.

Then

$$\int_{\Lambda} g(z)\, e^{f(z)}\, dz = O\, |\eta_0|\, (U(\eta_0))^{-1/n}\, |g(\eta_0)\, e^{f(\eta_0)}|\,. \tag{19}$$

Remark 1: In some cases the integral has the same order of magnitude as the function occurring behind the O sign, so that then the result can not be improved. For instance, if

$$\eta_0 = 1 \quad ; \quad g(z) = 1 \quad ; \quad f(z) = -\,\omega\,(z-1)^n,$$

where ω denotes a large positive number, then the halfline $(1\,,\infty)$ is a curve of steepest descent for the function $e^{f(z)}$ and we have for suitably chosen constant c

$$U(z) = \omega\,(z+1)^n;\ j = c\omega^{-1/n};\ g(\eta_0)\, e^{f(\eta_0)} = 1\,;\ \eta_0\,(U(\eta_0))^{-1/n} = 2^{-1}\,\omega^{-1/n}.$$

Consequently the exponent $-1/n$ occurring on the right hand side of (19) may not be replaced by a smaller number.

Remark 2: In many cases it is not necessary to know anything about the form of the integration path in order to verify the condition that $U(z)$ tends for each point z on Λ, uniformly in z, to infinity as $|\omega| \to \infty$. For instance, putting $\operatorname{Re} f(\eta_0) = u$ and $\operatorname{Im} f(\eta_0) = v$ we find that this condition is certainly satisfied if

$$u \sin\lambda < |v| \cos\lambda \tag{20}$$

and

$$\min(|v|\cos\lambda - u\sin\lambda\,,\ |v|\cos\lambda + u\sin\lambda) + \max(0\,, -|v|\sin\lambda - u\cos\lambda) \to \infty, \tag{21}$$

as $|\omega| \to \infty$.

If Λ is a curve of steepest descent for the function $e^{f(z)}$, then $\lambda = 0$ and the condition $U(z) \to \infty$, uniformly in z, is therefore certainly satisfied if

$$|\operatorname{Im} f(\eta_0)| + \max(0\,, -\operatorname{Re} f(\eta_0)) \to \infty \quad \text{as} \quad |\omega| \to \infty. \tag{22}$$

To prove this result it is sufficient to show that the left hand of (21) is $\leq \dfrac{1}{\sqrt{2}}\, |f(z)|$ for each point z on Λ, since in this case $|f(z)|$ and therefore certainly the function $U(z)$ defined by (16), tends to infinity, uniformly in z, as $|\omega| \to \infty$. The proof runs as follows.

Since the integration path Λ is a curve of descent with angle λ for the function $e^{f(z)}$ each point z of Λ has the property that $f(z)$ lies in the sector S with vertex $f(\eta_0) = u + iv$ defined by

$$-\lambda \leqq \arg(f(\eta_0) - w) \leqq \lambda .$$

It follows from (20) that the origin does not lie in this sector. Let us now distinguish three cases.

(1) The origin lies outside the sector

$$-\frac{\pi}{2} - \lambda \leqq \arg(f(\eta_0) - w) \leqq \lambda + \frac{\pi}{2} .$$

Then the distance from the origin to the sector S is equal to the distance from the origin to the vertex $\eta_0 = u + iv$, so that

$$|f(z)| \geqq |f(\eta_0)| = \sqrt{u^2 + v^2} .$$

The left hand side of (21) is at most equal to

$$\begin{aligned} & \big|\, |v| \cos\lambda - u \sin\lambda \big| + \big|\, |v| \sin\lambda + u \cos\lambda \big| \\ & \leqq \frac{1}{\sqrt{2}} \{(|v| \cos\lambda - u \sin\lambda)^2 + (|v| \sin\lambda + u \cos\lambda)^2\}^{1/2} \\ & = \frac{1}{\sqrt{2}} (u^2 + v^2)^{1/2} \leqq \frac{1}{\sqrt{2}} |f(z)| , \end{aligned}$$

which gives the required result.

(2) The origin lies in the sector

$$\lambda < \arg(f(\eta_0) - w) \leqq \lambda + \frac{\pi}{2} .$$

The distance from the origin to the sector S is equal to the distance from the origin to the line through $u + iv$ which makes an angle λ with the positive real axis. This distance is equal to $|v \cos\lambda - u \sin\lambda|$. Consequently $|f(z)|$ is $\geqq |v \cos\lambda - u \sin\lambda|$ and therefore certainly greater than or equal to the left hand side of (21). This gives the required result.

(3) The origin lies in the sector

$$-\lambda - \frac{\pi}{2} \leqq \arg(f(\eta_0) - w) < -\lambda .$$

The reasoning is the same as in case (2), with $|v \cos\lambda - u \sin\lambda|$ replaced by $|v \cos\lambda + u \sin\lambda|$. This completes the proof.

Remark 3: If all the exponents $\alpha_1, \dots, \alpha_n$ are positive, then the condition that $U(z)$ tends for each point z on Λ, uniformly in z, to infinity as $|\omega| \to \infty$ is certainly satisfied if the distance d from the origin to the integration path satisfies for large $|\omega|$ the order relation

$$\frac{1}{d} = o\left(\sum_{k=1}^{n} |u_k|^{1/\alpha_k}\right). \tag{23}$$

Indeed, if (23) holds, then for each $q > 1$ and for sufficiently large $|\omega|$ we find at least one positive integer $k \leqq n$ with

$$d^{-1} < q^{-1} |u_k|^{1/\alpha_k}, \quad \text{hence} \quad q^{\alpha_k} < |u_k| d^{\alpha_k},$$

so that for each point z on Λ

$$q^{\min(\alpha_1, \dots, \alpha_n)} < \sum_{k=1}^{n} |u_k z^{\alpha_k}| = U(z).$$

Consequently $U(z) \to \infty$, uniformly in z, as $|\omega| \to \infty$. In the same way we prove

Remark 4: If all the exponents $\alpha_1, \dots, \alpha_n$ are negative, then the condition that $U(z)$ tends for each point z on Λ, uniformly in z, to infinity as $|\omega| \to \infty$ is certainly satisfied if the distance from the origin to the integration path is

$$o\left(\sum_{k=1}^{n} |u_k|^{-(1/\alpha_k)}\right).$$

Remark 5: If the system $(\alpha_1, \dots, \alpha_n)$ contains at least one positive and at least one negative number, then the condition $U(z) \to \infty$, uniformly in z, is certainly satisfied if

$$\sum_{k,l} |u_k|^{-\alpha_l/(\alpha_k - \alpha_l)} |u_l|^{\alpha_k/(\alpha_k - \alpha_l)} \to \infty \quad \text{as} \quad |\omega| \to \infty;$$

the sum $\sum\limits_{k,l}$ is extended over the positive integers $k \leqq n$ and $l \leqq n$ with $\alpha_k \geqq 0$ and $\alpha_l < 0$.

Indeed, if $\alpha_k \geqq 0$ and $\alpha_l < 0$, then we have for each point $z \neq 0$

$$|u_k z^{\alpha_k}| + |u_l z^{\alpha_l}| \geqq |u_k|^{-\alpha_l/(\alpha_k - \alpha_l)} |u_l|^{\alpha_k/(\alpha_k - \alpha_l)},$$

so that

$$U(z) \geqq N^{-1} \sum_{k,l} |u_k|^{-\alpha_l/(\alpha_k - \alpha_l)} |u_l|^{\alpha_k/(\alpha_k - \alpha_l)},$$

where N denotes the number of the terms of the sum $\sum_{k,l}$. This completes the proof.

Remark 6: Later I shall prove Theorem 3 as an application of Theorem 4, but I now shall use Theorem 2 to prove Theorem 3 in the following weaker form, where condition (2) is replaced by the sharper condition:

(2*) Assume for each point z on Λ and each fixed positive number K that $g(w)$ denotes in the circle

$$|w-z| < K|z|(U(z))^{-1/n} \tag{24}$$

an analytic function of w, which satisfies for large $|\omega|$ the order relation (18) uniformly in z and w.

Proof of Theorem 3 in the weaker form.

We have for each $z \neq 0$ and for $h = 1, 2, \ldots, n$

$$z^h f^{(h)}(z) = \sum_{k=1}^{n} \alpha_k(\alpha_k - 1)\ldots(\alpha_k + 1 - h)\, u_k z^{\alpha_k}.$$

We consider this as a system of n linear equations with the unknown $u_k z^{\alpha_k}$. The determinant of this system is formed by the fixed numbers

$$\alpha_k(\alpha_k - 1)\ldots(\alpha_k + 1 - h) \quad (h = 1, \ldots, n\ ;\ k = 1, \ldots, n)$$

and is equal to the determinant of Vandermonde

$$\begin{vmatrix} \alpha_1 & \alpha_2 \ldots \alpha_n \\ \alpha_1^2 & \alpha_2^2 \ldots \alpha_n^2 \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ \alpha_1^n & \alpha_2^n \ldots \alpha_n^n \end{vmatrix}$$

and therefore $\neq 0$. Consequently for each positive integer $k \leq n$ we can write $u_k z^{\alpha_k}$ is a linear combination, with fixed coefficients, of $z^h f^{(h)}(z)$ $(h = 1, 2, \ldots, n)$. There exists therefore a fixed positive number c_1 such that for each $z \neq 0$ it is possible to find at least one positive integer $h \leq n$ with

$$U(z) = \sum_{k=1}^{n} |u_k z^{\alpha_k}| \leq c_1^h |z|^h |f^{(h)}(z)|.$$

By hypothesis $U(z)$ tends, uniformly in z, to infinity as $|\omega| \to \infty$, so that $U(z) \geq 1$ for sufficiently large $|\omega|$, hence

$$(U(z))^{1/n} \leqq (U(z))^{1/h} \leqq c_1 |z| \, | f^{(h)}(z) |^{1/h} .$$

In this way we find for each point $z \neq 0$ on Λ

$$(U(z))^{1/n} \leqq c_1 |z| \sum_{h=1}^{n} | f^{(h)}(z) |^{1/h} . \tag{25}$$

The origin does not lie on the integration path. Indeed, if all the numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ are positive integers, then the left hand side of (25) tends to zero as the point $z \neq 0$ tends on Λ to the origin, contrary to the hypothesis that $U(z)$ tends for $|\omega| \to \infty$ to infinity, uniformly in z. If not all the numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ are positive integers, then $f(z)$ is not analytic at $z = 0$, so that the origin does not lie on the integration path. Consequently (25) holds at each point z of the path.

Now we shall prove for sufficiently large $|\omega|$ that under the conditions of Theorem 3 in the weaker form certainly the conditions of Theorem 2 are satisfied with

$$l = n \quad \text{and} \quad \tau(z) = c_1^{-1} |z|^{-1} (U(z))^{1/n} .$$

Inequality (12) follows from (25). From $U(z) \to \infty$ it follows that $\sigma^{-1}(z) = o(z)$, uniformly in z, so that the points w lying in the circle (11) satisfy the order relation $w = (1 + o(1))z$. We have for $k = 1, 2, \dots, n$

$$| u_k w^k | = | u_k (1 + o(1)) z^k | = (1 + o(1)) \, | u_k z^k | ,$$

hence

$$U(w) = (1 + o(1)) \, U(z) ,$$

consequently

$$|w|^{-1} (U(w))^{1/n} = (1 + o(1)) \, |z|^{-1} (U(z))^{1/n} ,$$

therefore

$$\tau(w) = (1 + o(1)) \, \tau(z) ,$$

so that for sufficiently large $|\omega|$ the inequality

$$\tau(w) \geqq (1 - \gamma \cos \lambda) \, \tau(z)$$

holds. Furthermore

$$\begin{aligned} | f^{(n+1)}(w) | &= \left| \sum_{k=1}^{n} \alpha_k (\alpha_k - 1) \dots (\alpha_k - n) \, u_k w^{\alpha_k - n - 1} \right| \\ &= O\left(\sum_{k=1}^{n} | u_k w^{\alpha_k - n - 1} | \right) = O\left(\sum_{k=1}^{n} | u_k z^{\alpha_k - n - 1} | \right) \\ &= O\left(|z|^{-n-1} \right) U(z) = o\left(|z|^{-n-1} \right) (U(z))^{n+1} , \end{aligned}$$

so that for sufficiently large $|\omega|$

$$|f^{(n+1)}(w)| \leqq \sigma^{n+1}(z).$$

Up till now we have used neither Condition (2) nor (2*), but we need Condition (2*) for the verification of the second inequality (14). It follows from (25) that

$$\sigma^{-1}(z) = \Big(\sum_{h=1}^{n} |f^{(h)}(z)|^{1/h}\Big)^{-1} \leqq c_1^{-1}|z|(U(z))^{1/n},$$

According to Condition (2*), applied with $K = c_1^{-1}$, the function $g(w)$ is therefore an analytic function of w in the circle (11) with the property

$$g(w) = (1 + o(1))\, g(z),$$

so that for sufficiently large $|\omega|$ the second inequality (14) holds. Consequently all the conditions of Theorem 2 are satisfied. This yields Theorem 3 in the weaker form.

It is true that Theorem 2 is very general and sometimes gives results which can not be improved but it has some disadvantages, as we have already seen in Theorem 3 which can be proved only in the weak form by means of Theorem 2. It may happen that the integration path contains a point z with the property that at least one of the two functions $f(w)$ and $g(w)$ is not analytic in the circle $|w - z| < \sigma^{-1}(z)$. We need therefore a result similar to Theorem 2 in which $\sigma(z)$ is replaced by a larger function of z. This we shall do in Theorem 4 which contains Theorem 2 as a special case. It may also happen that at certain points z on Λ with $l = 1$ the condition

$$|f'(w)| \leqq |f'(z)|^2, \tag{26}$$

which is required in (14), is not satisfied. Then Theorem 2 may not be applied, but we shall see in Theorem 5 that under certain circumstances condition (26) is superfluous for the points z on Λ with $l = 1$. Theorem 5 contains Theorem 4 (and therefore also Theorem 2) as special cases, and it contains also, under a certain restriction, Theorem 1 as a special case.

Theorem 4: *For each positive integer n it is possible to find two positive numbers γ and c depending only on n such that the inequality*

$$\Big|\int_{\Lambda} g(z)\, e^{f(z)}\, dz\Big| < \frac{c\rho(\eta_0)}{\tau(\eta_0)\cos\lambda}\, |g(\eta_0)\, e^{f(\eta_0)}|, \tag{27}$$

where the path Λ is a curve of descent for $e^{f(z)}$ with angle $\lambda \left(0 \leqq \lambda < \frac{\pi}{2}\right)$ and initial point η_0, certainly holds if each point z on Λ satisfies the following

Condition: It is possible to find at least one positive integer $l \leqq n$ (l may depend on z) with the property that $f(w)$ and $g(w)$ are analytic functions of w in the circle

$$(28) \qquad |w-z| < \sigma^{-1}(z),$$

where

$$(29) \qquad \sigma(z) = \sum_{h=1}^{l} |\rho(z) f^{(h)}(z)|^{1/h} \geqq \tau(z) > 0 \quad \text{and} \quad \rho(z) \geqq 1$$

such that each point w which lies on Λ with (28) satisfies the inequalities

$$(30) \qquad \tau(w) \geqq \left(1 - \frac{\gamma \cos \lambda}{\rho(z)}\right) \tau(z) \quad \text{and} \quad \rho(w) \leqq \rho(z) + \gamma \cos \lambda$$

and that each point w with (28) satisfies the inequalities

$$(31) \qquad |f^{(l+1)}(w)| \leqq \sigma^{l+1}(w) \quad \text{and} \quad |g(w)| \leqq \left(1 + \frac{\gamma \cos \lambda}{\rho(z)}\right) |g(z)|.$$

Remark 1: The special case of this theorem with $\rho(z) = 1$ is identical with Theorem 2.

Remark 2: Above we have proved Theorem 3 in the weaker form by means of Theorem 2, but we have promised to prove Theorem 3 in the original form as an application of Theorem 4. The proof runs exactly as in the proof of Theorem 3 in the weak form up till the last paragraph which begins with "Up till now we have". This last paragraph can not be used here, since it applies Condition (2*). But we shall show that the conditions of Theorem 4 are satisfied with $l = n$ and $\rho(z) = \rho = \max(1, c_1^n)$; then $\rho(z)$ is independent of ω and z. Consequently the second inequality (30) is certainly satisfied, so that we have only to show that the second inequality (31) holds. We have by (29) and $\rho \geqq 1$

$$\sigma(z) \geqq \rho^{1/n} \sum_{h=1}^{n} |f^{(h)}(z)|^{1/h} \geqq c_1 \sum_{h=1}^{n} |f^{(h)}(z)|^{1/h} \geqq |z|^{-1} (U(z))^{1/n}$$

according to (25). Consequently each point w lying in the circle (28) lies in the circle

$$|w - z| < |z| (U(z))^{-1/n}$$

and satisfies therefore (18), which yields the required second inequality (31) for sufficiently large $|\omega|$. In this way we see that all the conditions of Theorem 4 are satisfied, so that Theorem 3 in the original form is a special case of Theorem 4.

For the proof of Theorem 4 we need two lemmas. The first of these two lemmas is the basis for the proof of all the theorems occurring in this paper, apart from the simple proof given in Theorem 1.

Lemma 1: *For each positive integer n it is possible to find two positive numbers $a < 1$ and $b \leqq \frac{1}{4}$ depending only on n with the following properties:*

If l is a positive integer $\leqq n$, if $\rho > 0$, z complex and if $f(w)$ is an analytic function of w in the circle $|w - z| < \sigma^{-1}$, where

$$\sigma = \sum_{h=1}^{l} \rho^{1/h} |f^{(h)}(z)|^{1/h} > 0, \tag{32}$$

in such a way that

$$|f^{(l+1)}(w)| \leqq \rho^{-1} \sigma^{l+1}, \tag{33}$$

then there exists at least one positive integer $k \leqq l$ such that the inequality

$$|f(w) - f(z)| \geqq \rho^{-1} b \tag{34}$$

holds for each point w with

$$|w - z| = l^{-1} k a \sigma^{-1}. \tag{35}$$

Proof: Without loss of generality we may assume that $\rho = 1$, for otherwise it is sufficient to replace f by $\rho^{-1} f$.

Let a and b denote two positive numbers with the property that it is possible to find l points $w_1, \ldots, w_l$ with

$$|w_k - z| = l^{-1} k a \sigma^{-1} \quad (k = 1, 2, \ldots, l) \tag{36}$$

and

$$|f(w_k) - f(z)| < b \quad (k = 1, 2, \ldots, l). \tag{37}$$

Then we have for $k = 1, 2, \ldots, l$

$$f(w_k) - f(z) = \sum_{h=1}^{l} \frac{f^{(h)}(z)}{h!} (w_k - z)^h + \frac{1}{l!} \int_z^{w_k} f^{(l+1)}(t) (w_k - t)^l \, dt.$$

The absolute value of the last term is according to (33) and (35) in absolute value

$$\leq \frac{\sigma^{l+1}}{(l+1)!}\,|\,w_k - z\,|^{l+1} \leq \frac{a^{l+1}}{(l+1)!}. \tag{38}$$

In this way we find for $k = 1, 2, \ldots, l$

$$\sum_{h=1}^{l} \frac{f^{(h)}(z)}{h!}\, a^h\, \sigma^{-h}\, t_k^h = u_k, \tag{39}$$

where $|t_k| = l^{-1} k$ and where, according to (37) and (38)

$$|\,u_k\,| < b + \frac{a^{l+1}}{(l+1)!}. \tag{40}$$

We may consider (39) as a system of l linear equations with the unknown $\frac{1}{h!}\, f^{(h)}(z)\, a^h\, \sigma^{-h}$ $(h = 1, 2, \ldots, l)$ and with the coefficients t_k^h. These coefficients form a determinant of Wronski with the value

$$\triangle = \prod_{1 \leq r < s \leq l} (t_s - t_r), \quad \text{where} \quad |\,t_s\,| = \frac{s}{l}.$$

Consequently $|\,t_s - t_r\,| \geq \dfrac{s-r}{l}$, so that

$$|\triangle| \geq \prod_{1 \leq r < s \leq l} \frac{s-r}{l},$$

where the right hand side denotes a positive number depending only on l. Each minor in the considered determinant of Wronski is in absolute value less than or equal to a suitably chosen number depending on l. Consequently we find for $h = 1, 2, \ldots, l$

$$|\,f^{(h)}(z)\, a^h\, \sigma^{-h}\,| < \frac{1}{2}\, c_l \left(b + \frac{a^{l+1}}{(l+1)!}\right), \tag{41}$$

where c_l denotes a suitably chosen number depending only on l.

It follows from the definition of σ given in (32) that

$$|\,f^{(h)}(z)\,|^{1/h} \geq l^{-1}\, \sigma$$

for at least one positive integer $h \leq l$, so that the left hand side of (41) is $\geq l^{-h}\, a^h$. For this integer h we find

$$l^{-h}\, a^h < \frac{1}{2}\, c_l \left(b + \frac{a^{l+1}}{(l+1)!}\right). \tag{42}$$

If we choose the positive numbers $a<1$ and $b \leqq \frac{1}{4}$ depending only on n so small that

$$a^{l+1-h} \leqq (l+1)!\, l^{-h} c_l^{-1} \quad \text{and} \quad b \leqq l^{-h} a^h c_l^{-1} \quad (1 \leqq h \leqq l \leqq n),$$

then

$$l^{-h} a^h \geqq c_l a^{l+1} / (l+1)! \quad \text{and} \quad \geqq c_l b,$$

so that the inequality (42) does not hold. This means that for this choice of a and b it is not possible to find l points $w_1, \dots, w_l$ with (36) and (37), so that there exists at least one positive integer $k \leqq l$ with the property that each point w with (35) satisfies (34). This completes the proof.

Lemma 2: If $0 \leqq \vartheta < 1$ and if $p_0, p_1, \dots$ denote positive numbers with

$$p_{h+1} \leqq p_h + \vartheta \qquad (h = 0, 1, \dots),$$

then

$$\sum_{h=0}^{\infty} \exp\left(-\left(\frac{1}{p_0} + \frac{1}{p_1} + \dots + \frac{1}{p_{h-1}}\right)\right) \leqq 1 + \frac{p_0}{1-\vartheta}. \tag{43}$$

Remark: We use this lemma in the proof of Theorem 4 and therefore also in the proof of the particular case formulated in Theorem 2; in the special case however that in Theorem 4 $\rho(z)$ is independent of z (this is indeed the case in Theorem 2 where $\rho(z) = 1$) we need this Lemma 2 only in the case that p_h is independent of h and then the lemma is obvious, since the left hand side of (43) is

$$\sum_{h=0}^{\infty} e^{-(h/p_0)} = 1 + \frac{1}{e^{1/p_0} - 1} \leqq 1 + p_0.$$

Proof: In the proof I may assume that ϑ is positive. We have $\log(1+u) \leqq u$ for each $u \geqq 0$, so that for each integer $k \geqq 0$

$$\log \frac{p_0 + (k+1)\vartheta}{p_0 + k\vartheta} \leqq \frac{\vartheta}{p_0 + k\vartheta}. \tag{44}$$

Consequently we have for each positive integer h

$$\log \frac{p_0 + h\vartheta}{p_0} = \sum_{k=0}^{h-1} \log \frac{p_0 + (k+1)\vartheta}{p_0 + k\vartheta} \leqq \vartheta \sum_{k=0}^{h-1} \frac{1}{p_0 + k\vartheta}.$$

Putting $\vartheta^{-1} = \eta$ we find therefore

$$(45)\quad \begin{cases} \displaystyle\sum_{h=0}^{\infty} \exp\left(-\left(\frac{1}{p_0}+\frac{1}{p_1}+\ldots+\frac{1}{p_{h-1}}\right)\right) = 1+\sum_{h=1}^{\infty}\exp\left(-\left(\frac{1}{p_0}+\ldots+\frac{1}{p_{h-1}}\right)\right) \\ \displaystyle\leqq 1+\sum_{h=1}^{\infty}\exp\left(-\sum_{k=0}^{h-1}\frac{1}{p_0+k\vartheta}\right) \leqq 1+\sum_{h=1}^{\infty}\exp\left(-\eta\log\frac{p_0+h\vartheta}{p_0}\right). \end{cases}$$

For each positive integer h we have in the interval $h-1 \leqq u \leqq h$

$$(p_0+h\vartheta)^{-\eta} \leqq (p_0+u\vartheta)^{-\eta}$$

so that

$$(p+h\vartheta)^{-\eta} \leqq \int_{h-1}^{h} (p_0+hu)^{-\eta}\,du\,,$$

hence

$$\sum_{h=1}^{\infty}(p+h\vartheta)^{-\eta} \leqq \int_0^{\infty}(p+u\vartheta)^{-\eta}\,du = \frac{p_0^{1-\eta}}{(\eta-1)\,\vartheta} = \frac{p_0^{1-\eta}}{1-\vartheta}\,.$$

Consequently the required inequality (43) follows from (45).

Now I proceed to the proof of Theorem 4 which I formulate in such a way that it can be also used as a part of the proof of Theorem 5, the last theorem occurring in this paper.

First step. Choice of the successor.

Let z denote an arbitrary point on Λ and let l denote the smallest positive integer $l \leqq n$ for which the condition occurring in Theorem 4 is satisfied; this integer l may depend on z and is uniquely defined by z. Let k denote the smallest positive integer $\leqq l$ which possesses the property mentioned in Lemma 1; this integer is uniquely defined by z; in this proof a and b denote the positive numbers depending only on n occurring in this Lemma 1.

I call the successor of z the first point w lying on Λ behind z with

$$(46)\qquad |w-z| = l^{-1}k\,\alpha\,\sigma^{-1}(z)$$

if such a point w exists; otherwise I call the endpoint of Λ the successor of z.

From this definition it follows that each point z on Λ possesses one and only one successor.

Second step: We choose $\gamma = \frac{b}{24}$. If z is a point on Λ whose successor w does not coincide with the endpoint of Λ, then

$$\tau^{-1}(w)\,|\,g(w)\,e^{f(w)}\,| < \tau^{-1}(z)\,|\,g(z)\,e^{f(z)}\,|\exp\left(-\frac{12\gamma\cos\lambda}{\rho(z)}\right). \tag{47}$$

Proof: Since Λ is a curve of descent with angle λ for the function $e^{f(z)}$ we have

$$\mathrm{Re}\,(f(w) - f(z)) \leqq -(\cos\lambda)\,|\,f(w) - f(z)\,| \leqq -\frac{b\cos\lambda}{\rho(z)}$$

according to Lemma 1. Formula (31) gives

$$\left|\frac{g(w)}{g(z)}\right| \leqq 1 + \frac{\gamma\cos\lambda}{\rho(z)} < e^{\gamma\cos\lambda/\rho(z)}$$

and the first inequality (30) yields

$$\frac{\tau(z)}{\tau(w)} \leqq \left(1 - \frac{\gamma\cos\lambda}{\rho(z)}\right)^{-1} < e^{11\gamma\cos\lambda/\rho(z)}.$$

This gives the required inequality (47).

Third step: If z is a point on Λ whose successor w may coincide with the endpoint of Λ, then

$$\left|\int_z^w g(t)\,e^{f(t)}\,dt\right| < 2e^e\,\sigma^{-1}(z)\,|\,g(z)\,e^{f(z)}\,|. \tag{48}$$

Proof: By the definition of the successor w of z on the integration path each point t of the arc $(z\,,w)$ satisfies in virtue of $a<1$ and $k \leqq l$ the inequality

$$|\,t - z\,| < \sigma^{-1}(z).$$

In this circle $f(t)$ and $g(t)$ are by hypothesis analytic functions of t, so that the integral of $g(t)\,e^{f(t)}$ along the arc $(z\,,w)$ is equal to the integral along the line segment $(z\,,w)$. At each point t of that line segment we have

$$f(t) - f(z) = \sum_{r=1}^{l}\frac{f^{(r)}(z)}{r\,!}(t-z)^r + \frac{1}{l\,!}\int_z^t f^{(l+1)}(u)\,(t-u)^l\,du. \tag{49}$$

According to the definition of $\sigma(z)$ given in (29) we have for $1 \leqq r \leqq l$

$$|\,f^{(r)}(z)\,|^{1/r} \leqq (\rho(z))^{-(1/r)}\,\sigma(z) \leqq \sigma(z),$$

hence

$$\left|\sum_{r=1}^{l}\frac{f^{(r)}(z)}{r\,!}(t-z)^r\right| \leqq \sum_{r=1}^{\infty}\frac{1}{r\,!} = e - 1.$$

Moreover it follows from the first of the inequalities (31) that the last term in (49) is in absolute value

$$\leqq (\sigma(z))^{l+1}(\sigma(z))^{-l-1} = 1 .$$

By means of (49) we find therefore for each point t of the line segment (z, w)

$$|f(t) - f(z)| \leqq e .$$

Furthermore it follows from the second of the inequalities (31)

$$|g(t)| \leqq \left(1 + \frac{\gamma \cos \lambda}{\rho(z)}\right) |g(z)| < 2|g(z)| ,$$

so that the left hand side in (48) is

$$< 2e^e |g(z) e^{f(z)}| |w - z| < 2e^e |g(z) e^{f(z)}| \sigma^{-1}(z) .$$

In order to simplify the proof of Theorem 5 I apply in the last step the result of the second step only in the weaker form that the factor 12 occurring in the exponent on the right hand side is replaced by 11, I use the result of the third step only in the weaker form that the factor $2e^e$ occurring on the right hand side is replaced by $2e^e + 1$ and finally I apply the second inequality (30) only in the weaker form

(50) $$\rho(w) < \rho(z) + 10\gamma \cos \lambda .$$

Fourth step: End of the proof.

Consider the points $\eta_0, \eta_1, \ldots,$ where η_0 is the initial point of Λ and where η_{h+1} $(h \geqq 0)$ is the successor of η_h. If the number $H+2$ of these numbers is finite, then η_{H+1} is the endpoint of Λ and we have

(51) $$\int_{\Lambda} g(t) e^{f(t)} dt = \sum_{h=0}^{H} \int_{\eta_h}^{\eta_{h+1}} g(t) e^{f(t)} dt .$$

To show that this formula also holds for $H = \infty$ I deduce from (47) (applied with the factor 11 instead of 12) for $h = 0, 1, \ldots$

(52) $$\tau^{-1}(\eta_h) |g(\eta_h) e^{f(\eta_h)}| \leqq \tau^{-1}(\eta_0) |g(\eta_0) e^{f(\eta_0)}| \exp\left(-11\gamma \cos \lambda\right) \sum_{r=0}^{h-1} \frac{1}{\rho(\eta_r)} .$$

The function

(53) $$p(z) = \frac{\rho(z)}{11\gamma \cos \lambda}$$

satisfies according to (50) for $h = 0, 1, \ldots$ the inequality

(54) $$p(\eta_{h+1}) - p(\eta_h) < \frac{10}{11},$$

so that according to Lemma 2 the series $\frac{1}{\rho(\eta_0)} + \frac{1}{\rho(\eta_1)} + \ldots$ diverges. Consequently the left hand side of (52) tends for $h \to \infty$ to zero, so that η_h tends for $h \to \infty$ to the endpoint of Λ; this yields (51) in the case $H = \infty$.

Application of the result obtained in the third step gives therefore

$$\left| \int_\Lambda g(t)\, e^{f(t)}\, dt \right| < (2e^e + 1) \sum_{h=0}^{H} \tau^{-1}(\eta_h) \left| g(\eta_h)\, e^{f(\eta_h)} \right|$$

$$\leqq (2e^e + 1)\, \tau^{-1}(\eta_0) \left| g(\eta_0)\, e^{f(\eta_0)} \right| \sum_{h=0}^{H} \left| \exp\left(-\sum_{r=0}^{h-1} \frac{1}{p(\eta_r)} \right) \right|,$$

where the last sum is, according to Lemma 2 applied with $\vartheta = \frac{10}{11}$, at most equal to

$$1 + 11 p(\eta_0) = 1 + \frac{10\, \rho(\eta_0)}{b \cos \lambda} < \frac{11\, \rho(\eta_0)}{b \cos \lambda}.$$

This completes the proof.

Theorem 5: *For each positive integer n it is possible to find two positive numbers γ and c depending only on n such that the inequality*

(55) $$\left| \int_\Lambda g(z)\, e^{f(z)}\, dz \right| < \frac{c\, \rho(\eta_0)}{\tau(\eta_0) \cos \lambda} \left| g(\eta_0)\, e^{f(\eta_0)} \right|,$$

where the integration path is a curve of descent of $e^{f(z)}$ with angle λ $\left(0 \leqq \lambda < \frac{\pi}{2}\right)$ and initial point η_0, certainly holds if each point of z which does not satisfy the condition formulated in Theorem 4 satisfies the following

Condition: *The functions $g(t) \neq 0$, $\rho(t) \geqq 1$ and $\tau(t) > 0$ are defined and differentiable along Λ at the point $t = z$ and $f(t)$ is continuously differentiable along Λ at $t = z$ in such a way that*

(56) $$\sigma(z) = \rho(z) \left| f'(z) \right| \geqq \tau(z),$$

$$\left|\frac{g'(z)}{g(z)}\right| + \left|\frac{\tau'(z)}{\tau(z)}\right| \leq \frac{1}{2}\,|f'(z)|\cos\lambda \tag{57}$$

and

$$\left|\frac{\rho'(z)}{\rho(z)}\right| \leq \frac{1}{3}\,|f'(z)|\cos\lambda\,. \tag{58}$$

Remark: I call a point z on Λ a point of the first kind if it satisfies the condition formulated in Theorem 5. I call a point z on Λ a point of the second kind if it satisfies the condition formulated in Theorem 4. In Theorem 5 each point z on Λ belongs to the first or to the second or to both kinds.

Theorem 4 is the special case of Theorem 5 in which each point of the integration path belongs to the second kind. Theorem 1 is the special case of Theorem 5 with $\rho(z)=1$, where each point of Λ belongs to the first kind, apart from two facts: (1) in Theorem 1 the assertion involves the factor $2e$, whereas the assertion of Theorem 5 involves a suitably chosen constant c. (2) the condition of continuity imposed on $f'(t)$ in Theorem 5 does not occur in Theorem 1.

Proof of Theorem 5.

First step: Choice of the successor.

For each point of the second kind we define the successor in the same way as in the first step of the proof of Theorem 4. Let us now consider a point z on Λ which does not belong to the second kind and is therefore a point of the first kind. Let w denote the point lying on Λ behind z with

$$\int_z^w |f'(t)|\,|dt| = \frac{b}{\rho(z)}\,, \tag{59}$$

if such a point w exists; otherwise w is the endpoint of Λ. In this proof a and b denote again the positive numbers depending on n with the properties mentioned in Lemma 1, where $\sigma=\sigma(z)$ and $\rho=\rho(z)$. If all the points of the open arc $(z\,,w)$ belong to the first kind, then I call w the successor of z. Otherwise I consider the last point z^* on the arc $(z\,,w)$ such that all the points of the open arc $(z\,,z^*)$ belong to the first kind. Consequently z^* coincides with z or lies on Λ between z and w.

Let us now show that z^* is a point of the second kind. Otherwise it would be a point of the first kind, which is a limit point of a set formed by points t of the second kind which lie on Λ behind z^*. Consequently there would exist a positive integer $l \leqq n$ such that z^* is a limit point of a set E formed by points t which lie on Λ behind z^* and satisfy the condition formulated in Theorem 4 with this particular value of l. According to our hypothesis z^* is a point of the first kind, so that by hypothesis $g(t)$, $\rho(t)$ and $\tau(t)$ are differentiable along Λ at $t = z^*$ and $f(t)$ is continuously differentiable along Λ at $t = z^*$. Consequently these four functions and also the function $\sigma(t)$ defined in (56) are continuous on Λ at $t = z^*$. Let us consider a point w lying in the circle

$$|w - z^*| < \sigma^{-1}(z^*).$$

This point w lies in the circle

$$|w - t| < \sigma^{-1}(t)$$

for each point t of E which lies close enough to z^*. From the fact that such a point t belongs to the second kind it follows that $f(w)$ and $g(w)$ are analytic functions of w such that the inequalities (30) and (31) hold with z replaced by t. Considerations of continuity show that the inequalities (30) and (31) are also valid with z replaced by z^*. Consequently z^* is a point of the second kind. By definition I call the successor of z^* also the successor of z.

From this definition it follows that each point on the integration path possesses one and only one successor.

Second step: We choose $\gamma = \dfrac{b}{24}$. If z is a point on Λ whose successor w does not coincide with the endpoint of Λ, then

$$\tau^{-1}(w)\,|g(w)\,e^{f(w)}| < \tau^{-1}(z)\,|g(z)\,e^{f(z)}|\exp\left(-\frac{11\,\gamma\cos\lambda}{\rho(z)}\right) \tag{60}$$

and

$$\rho(w) - \rho(z) < 10\,\gamma\cos\lambda. \tag{61}$$

Proof: If z is a point of the second kind, then (60) follows from the second step in the proof of Theorem 4 and we have by the second inequality (30)

$$\rho(w) - \rho(z) \leqq \gamma\cos\lambda < 10\,\gamma\cos\lambda. \tag{62}$$

If z is a point of the first kind I distinguish two cases:

[1] All the points of the open arc (z, w) belong to the first kind. Then

$$\operatorname{Re}(f(w)-f(z)) \leqq -(\cos\lambda)\int_z^w |f'(t)|\,|dt|.$$

It follows from (57) that

$$\log\left|\frac{\tau(z)g(w)}{\tau(w)g(z)}\right| \leqq \left|\log\frac{\tau(z)g(w)}{\tau(w)g(z)}\right| \leqq \int_z^w\left\{\left|\frac{\tau'(t)}{\tau(t)}\right|+\left|\frac{g'(t)}{g(t)}\right|\right\}|dt|$$

$$\leqq \frac{1}{2}(\cos\lambda)\int_z^w |f'(t)|\,|dt|,$$

so that

$$(63)\quad \log\left|\frac{\tau(z)g(w)}{\tau(w)g(z)}\right| + \operatorname{Re}(f(w)-f(z)) \leqq -\frac{1}{2}(\cos\lambda)\int_z^w |f'(t)|\,|dt|.$$

The right hand side is according to (59) at most equal to $-\dfrac{b\cos\lambda}{2\rho(z)}$. This gives the required result (60).

We have by (58)

$$\left|\log\frac{\rho(w)}{\rho(z)}\right| \leqq \int_z^w\left|\frac{\rho'(t)}{\rho(t)}\right||dt| \leqq \frac{1}{3}(\cos\lambda)\int_z^w |f'(t)|\,|dt|$$

$$\leqq \frac{b\cos\lambda}{3\rho(z)} = \frac{8\gamma\cos\lambda}{\rho(z)},$$

so that

$$(64)\qquad \frac{\rho(w)}{\rho(z)} \leqq \exp\left(\frac{8\gamma\cos\lambda}{\rho(z)}\right) < 1 + \frac{9\gamma\cos\lambda}{\rho(z)},$$

since $8\gamma = \dfrac{b}{3} < \dfrac{1}{9}$. In this way we find (61) even with the factor 9 instead of 10.

[2] Not all the points of the open arc (z, w) belong to the first kind. As we have seen in Step 1 the last point z^* on this arc with the property that all the points of the open arc (z, z^*) belong to the first kind is a

point of the second kind with successor w. According to Step 2 in the proof of Theorem 4 inequality (47) holds with z replaced by z^*. Formula (63) applied with z^* instead of w yields

$$\log \left| \frac{\tau(z)\, g(z^*)}{\tau(z^*)\, g(z)} \right| + \operatorname{Re}(f(z^*) - f(z)) \leqq 0 .$$

Combining these two results we obtain

$$(65) \qquad \begin{cases} \tau^{-1}(w) \left| g(w)\, e^{f(w)} \right| \leqq e^{-(b \cos\lambda / 2\rho(z^*))}\, \tau^{-1}(z^*) \left| g(z^*)\, e^{f(z^*)} \right| \\ \qquad\qquad \leqq e^{-(b \cos\lambda / 2\rho(z^*))}\, \tau^{-1}(z) \left| g(z)\, e^{f(z)} \right| . \end{cases}$$

We have according to (58)

$$\left| \log \frac{\rho(z^*)}{\rho(z)} \right| \leqq \int_z^{z^*} \left| \frac{\rho'(t)}{\rho(t)} \right| |dt| \leqq \frac{1}{3} (\cos\lambda) \int_z^{z^*} |f'(t)|\, |dt|$$

$$\leqq \frac{b \cos\lambda}{3\rho(z)} < -\log\left(1 - \frac{1}{12}\right),$$

since $b \leqq \frac{1}{4}$ and $\rho(z) \geqq 1$, hence

$$\rho(z^*) < \frac{12}{11}\, \rho(z) .$$

Consequently (65) yields the required inequality (60).

Applying (64) with w replaced by z^* and applying the second inequality (30) with z replaced by z^* we find by addition

$$\rho(w) - \rho(z) < 9\,\gamma\cos\lambda + \gamma\cos\lambda = 10\,\gamma\cos\lambda .$$

This completes the proof.

Third step: If z is a point on Λ whose successor w may coincide with the endpoint of Λ, then

$$(66) \qquad \left| \int_z^w g(t)\, e^{f(t)}\, dt \right| < (2e^e + 1)\, \tau^{-1}(z) \left| g(z)\, e^{f(z)} \right| .$$

Proof: In the case that z is a point of the second kind the third step in the proof of Theorem 4, in conjunction with $\tau(z) \leqq \sigma(z)$, gives the required result even with the factor $2e^e$ instead of $2e^e + 1$ on the right hand side. If z is a point of the first kind I distinguish two cases.

[1] All the points of the open arc (z, w) belong to the first kind. Replacing in (63) w by an arbitrary point t of the arc (z, w) we obtain

$$(67) \qquad \tau^{-1}(t) \left| g(t)\, e^{f(t)} \right| \leqq \tau^{-1}(z) \left| g(z)\, e^{f(z)} \right| ,$$

so that the left hand side of (66) is

$$\leq \tau^{-1}(z)\,|\,g(z)\,e^{f(z)}\,|\int_z^w \tau(t)\,|\,dt\,|\,. \tag{68}$$

Replacing in (64) w by t we obtain $\rho(t) < 2\rho(z)$ and according to (56) we have $\tau(t) \leq \rho(t)\,|\,f'(t)\,|$, so that the integral occurring in (68) is

$$\leq 2\rho(z)\int_z^w |\,f'(t)\,|\,|\,dt\,| \leq 2b < 1\,.$$

This gives the required inequality (66), even with the factor 1 instead of $2e^e + 1$ on the right hand side.

[2] Not all the points of the open arc $(z\,,w)$ belong to the first kind. Then the last point z^* on this arc with the property that all the points of the open arc $(z\,,z^*)$ belong to the first kind is a point of the second kind with successor w. According to the third step in the proof of Theorem 4 the contribution to the left hand side of (66) of the arc $(z^*\,,w)$ is less than

$$2e^e\,\tau^{-1}(z^*)\,|\,g(z^*)\,e^{f(z^*)}\,|$$

and according to [1] the contribution of the arc $(z\,,z^*)$ is less than

$$\tau^{-1}(z)\,|\,g(z)\,e^{f(z^*)}\,|\,.$$

This, in conjunction with (67), applied with $t = z^*$, yields the required result.

The fourth step, the end of the proof, runs in exactly the same way as in Theorem 4. This completes the proof of Theorem 5.

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DISTRIBUTIONS WITH COMPATIBLE NEUTRICES

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Section 1: The sum of compatible neutrices

In the definition of distributions given in my previous papers [1] on neutrix calculus I have restricted myself to neutrices with independent variables except in report 143 of the Mathematics Research Center at Madison where I have extended the domain of applicability of the neutrix calculus by admitting certain neutrices with dependent variables. This extension has the consequence that also in the theory of distributions the said neutrices with dependent variables may be used. This paper is devoted to the exposition of the theory of distributions based on this idea.

Let N be a neutrix with variable ξ, with domain N' and with negligible functions $\nu(\xi)$. The negligible functions are defined for each element ξ of the domain N' in such a way that $\nu(\xi)$ is an element of a given range; a range is a commutative additive group. That N is a neutrix means that it satisfies the neutrix condition: if a function $\nu(\xi)$ negligible in N is independent of ξ, then $\nu(\xi)$ is identically equal to zero.

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The previous papers on neutrix calculus are: Neutrices, J. Soc. Indust. Appl. Math., 7, No. 3, September, 1959, 253—279.

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Neutrix calculus III, General neutrix calculus, report 144 of the Mathematics Research Center, Madison.

Let P be a neutrix with a variable η, with domain P', with negligible functions $\pi(\eta)$ and with the same range as N. The variable ξ traverses N' and the variable η traverses P'. It may be that ξ and η are independent variables, but if this is not the case, then there exists at least one couple (ξ_0, η_0) formed by an element ξ_0 of N' and an element η_0 of P' which does not enter into consideration; this means that it is excluded that ξ assumes the value ξ_0 and η assumes simultaneously the value η_0.

The simultaneous use of N and P means that we may neglect each function $\nu(\xi)$ negligible in N and also each function $\pi(\eta)$ negligible in P, therefore also the sum $\nu(\xi) + \pi(\eta)$. The fundamental rule in neutrix calculus is that we never neglect a constant $\neq 0$. For this reason we call N and P compatible when they satisfy the following condition: if there exist a function $\nu(\xi)$ negligible in N and a function $\pi(\eta)$ negligible in P such that their sum $\nu(\xi) + \pi(\eta)$ is for each couple (ξ, η) which enters into consideration independent of ξ and η, then this sum is identically equal to zero.

Example 1: Two identical neutrices are compatible. Two neutrices with independent variables are compatible.

This is obvious.

Example 2: I call a neutrix P an enlargement of the neutrix N, if they have the same domain, the same variable and if each function negligible in N is also negligible in P. A neutrix is always compatible with each enlargement of itself.

Proof: Assume that for each element ξ of N'

$$\nu(\xi) + \pi(\xi) = \gamma,$$

where γ is independent of ξ, where $\nu(\xi)$ is negligible in N and $\pi(\xi)$ is negligible in P. Then the functions $\nu(\xi)$, $\pi(\xi)$ and $\nu(\xi) + \pi(\xi)$ are negligible in the neutrix P. The function $\nu(\xi) + \pi(\xi)$ is independent of ξ and therefore equal to zero according to the neutrix condition of P. This completes the proof.

Example 3: If N and P are compatible neutrices, then N is compatible with each neutrix R of which P is an enlargement.

Indeed, if the relation

$$\nu(\xi) + \rho(\eta) = \gamma$$

holds for all the systems (ξ, η) which enter into consideration, where $\nu(\xi)$ is negligible in N and $\rho(\eta)$ is negligible in R, then $\rho(\eta)$ is certainly negligible in P, so that the compatibility of N and P yields $\gamma = 0$.

Example 4: Let N and P be neutrices with variables ξ and η, such that the domain N' of N is a subset of the domain P' of P; for each η belonging to N' we put $\eta = \xi$. Finally we assume for each function $\pi(\eta)$ negligible in P that the function $\pi(\xi)$, where ξ traverses N', is negligible in N. Then N and P are compatible.

Remark: In short: two neutrices with the same negligible functions and with the property that the domain of one of the neutrices is a subset of the domain of the other neutrix, are compatible.

Proof: Assume that the relation

$$\nu(\xi) + \pi(\eta) = \gamma, \tag{1}$$

where γ is constant, holds for each couple (ξ, η) which enters into consideration. Then (1) holds certainly with $\eta = \xi$ for each element ξ of N'. Each of the functions $\nu(\xi)$, $\pi(\xi)$ and $\nu(\xi) + \pi(\xi)$ is negligible in N. The function $\nu(\xi) + \pi(\xi)$ is independent of ξ and therefore equal to zero according to the neutrix condition of N. This completes the proof.

Example 5: Let N and P be the neutrices treated in the preceding example. Each neutrix R which is compatible with N is certainly compatible with P.

Proof: If for each (η, ζ) which enters into consideration

$$\pi(\eta) + \rho(\zeta) = \gamma,$$

where $\pi(\eta)$ and $\rho(\zeta)$ are negligible respectively in P and R, then we have certainly for each couple (ξ, ζ) which enters into consideration and where ξ is an element of N'

$$\pi(\xi) + \rho(\zeta) = \gamma.$$

Here $\pi(\xi)$ is by hypothesis negligible in N, so that the compatibility of N and R yields $\gamma = 0$.

Let N and P be two compatible neutrices with variables ξ and η, with domains N' and P' and with negligible functions $\nu(\xi)$ and $\pi(\eta)$.

Let S' denote the set formed by all the couples (ξ, η) which enter into consideration. The functions $\nu(\xi) + \pi(\eta)$, where $\nu(\xi)$ is an arbitrary function negligible in N and $\pi(\eta)$ is an arbitrary function negligible in P, are defined for each element (ξ, η) of S'. From the fact that N and P are compatible it follows that the set formed by all these functions $\nu(\xi) + \pi(\eta)$ is a neutrix. I call this neutrix the sum $S = N + P$ of N and P. It is clear that this addition is commutative.

Let $N_1, N_2, \ldots, N_s$ be neutrices with variables $\xi_1, \xi_2, \ldots, \xi_s$, with domains $N'_1, N'_2, \ldots, N'_s$ and with negligible functions

$$\nu_1(\xi_1), \nu_2(\xi_2), \ldots, \nu_s(\xi_s).$$

If the variables $\xi_1, \ldots, \xi_s$ are not independent, then there exists at least one system $(\alpha_1, \ldots, \alpha_s)$, where α_σ belongs to N'_σ, which does not enter into consideration; this means that it is excluded that the variables $\xi_\sigma (\sigma = 1, \ldots, s)$ assume simultaneously the values α_σ. We call $N_1, \ldots, N_s$ compatible when they satisfy the following condition: if there exist functions $\nu_\sigma(\xi_\sigma)$ negligible in $N_\sigma (\sigma = 1, \ldots, s)$ such that for each system $(\xi_1, \ldots, \xi_s)$ which enters into consideration the sum $\sum_{\sigma=1}^{s} \nu_\sigma(\xi_\sigma)$ is independent of $\xi_1, \ldots, \xi_s$, then this sum is identically equal to zero.

If the s neutrices are compatible, we denote by S' the set formed by all the systems $(\xi_1, \ldots, \xi_s)$ which enter into consideration. The sums $\sum_{\sigma=1}^{s} \nu_\sigma(\xi_\sigma)$ mentioned above form a neutrix with domain S'. This neutrix is called the sum

$$S = N_1 + N_2 + \ldots + N_s \quad \text{of} \quad N_1, \ldots, N_s.$$

It is clear that the addition is not only commutative, but also associative:

$$(N_1 + N_2) + N_3 = N_1 + (N_2 + N_3)$$

denotes the neutrix whose domain is formed by the triplets (ξ_1, ξ_2, ξ_3) which enter into consideration and with the negligible functions

$$\nu_1(\xi_1) + \nu_2(\xi_2) + \nu_3(\xi_3).$$

Notice that we define the sum only for compatible neutrices. Notice furthermore that a function negligible in one of the neutrices N_h is negligible in the sum.

Example 6: The sum of s neutrices each equal to N, is equal to N.

Example 7: If $N_1, \ldots, N_s$ are neutrices with independent variables $\xi_1, \ldots, \xi_s$, with domains $N'_1, \ldots, N'_s$ and with negligible functions $\nu_1(\xi_1), \ldots, \nu_s(\xi_s)$, then the domain S' of the sum $S = \sum_{\sigma=1}^{s} N_\sigma$ is the direct product of the domains $N'_1, \ldots, N'_s$. The variable of S is $(\xi_1, \ldots, \xi_s)$, where the variables $\xi_\sigma (\sigma = 1, \ldots, s)$ traverse N'_s independently. Finally the negligible functions of S are the functions of the form $\sum_{\sigma=1}^{s} \nu_\sigma(\xi_\sigma)$.

Example 8: If the neutrix P is an enlargement of the neutrix N, then $N + P = P$.

Example 9: The domain of the sum S of the two neutrices N and P treated in the examples 4 and 5 is formed by the elements ξ of N' and the couples (ξ, η), where ξ is an arbitrary element of N' and η is an arbitrary element of P' outside N'. The negligible functions of S are the functions

$\nu(\xi) + \pi(\xi)$ for each ξ in N':

$\nu(\xi) + \pi(\eta)$ for each ξ in N' and each η in P' outside N'.

Let us now consider a set $\mathfrak{S}$ formed by infinitely many compatible neutrices. Then we define the sum S of all the neutrices belonging to $\mathfrak{S}$ in the following way. The domain S' of S is formed by all the systems $\xi = (\xi_1, \ldots, \xi_s)$, where s is an arbitrary positive integer and where $\xi_1, \ldots, \xi_s$ are the variables of neutrices $N_1, \ldots, N_s$ belonging to $\mathfrak{S}$. For each element $\xi = (\xi_1, \ldots, \xi_s)$ of S' I consider the functions $\sum_{\sigma=1}^{s} \nu_\sigma(\xi_\sigma)$, where $\nu_\sigma(\xi_\sigma)$ denotes an arbitrary function negligible in $N_\sigma (\sigma = 1, \ldots, s)$. These functions form a neutrix. Indeed, if for each element ξ of S'

(2) $$\sum_{\sigma=1}^{s} \nu_\sigma(\xi_\sigma) = \gamma,$$

where γ is independent of $\xi_1, \ldots, \xi_s$, then the left-hand side of (2) represents a function negligible in the neutrix $N_1 + \ldots + N_s$ which is for all the elements $\xi = (\xi_1, \ldots, \xi_s)$ of the domain of $N_1 + \ldots + N_s$ independent of ξ, so that this function is identically equal to zero. I call this neutrix the sum of all the neutrices belonging to $\mathfrak{S}$.

In this way we have defined the sum of a finite or infinite number of compatible neutrices.

Section 2: The basis

Introduce a certain set $\mathfrak{B}$ formed by a finite or infinite number of compatible neutrices. I suppose that this set, which I call a basis, is known. Restricting ourselves continually to neutrices which can be written as finite sums of neutrices belonging to the basis we build up a certain calculus. What happens if nevertheless a certain construction leads to a neutrix N which is compatible with all the neutrices occurring in the basis but which can not be written as a finite sum of neutrices belonging to the basis? Do we ignore this new neutrix? No, probably we construct a new basis $\mathfrak{B}^*$ formed by N and by all the neutrices belonging to $\mathfrak{B}$. Then, by means of this new basis, we construct a new calculus. We shall see that all the identities occurring in the original calculus with basis $\mathfrak{B}$ remain valid in the new calculus with basis $\mathfrak{B}^*$ (principle of permanence).

We obtain the calculus developed in my previous papers by imposing on the basis the condition that each neutrix occurring in the basis can be written as a finite sum of neutrices with independent variables.

In the applications the basis is often formed by only one neutrix. For instance in the theory of the distributions of Schwartz we restrict ourselves to only one neutrix, namely the neutrix named after Schwartz.

Section 3: Distributions

Let $\mathfrak{B}$ be a given basis and let B denote the sum of all the neutrices belonging to the basis. Then each function negligible in one of the neutrices occurring in the basis is negligible in B.

Let N be a neutrix with variable ξ which can be written as a finite sum of neutrices occurring in the basis. Let $f(\xi)$ be a function defined for each element ξ of the domain N' of N. The class d formed by all the functions of the form $f(\xi)+\nu(\xi)$, where $\nu(\xi)$ is an arbitrary function negligible in N, is called the distribution with neutrix N generated by $f(\xi)$. Each function occurring in d is a generating function of d.

Let P be a neutrix with variable η which also can be written as a finite sum of neutrices belonging to the basis. Let d_1 be the distribution with neutrix P generated by a given function $g(\eta)$. I call the distributions d and d_1 equal if and only if $f(\xi)-g(\eta)$ is negligible in the neutrix B. The notion of equality is independent of the choice of the generating functions $f(\xi)$ and $g(\eta)$, for if $f(\xi)$ and $f_1(\xi)$ are generating functions of d, whereas $g(\eta)$ and $g_1(\eta)$ are generating functions of d_1, then $f_1(\xi)-f(\xi)$ is negligible in N, therefore certainly in B, whereas $g_1(\eta)-g(\eta)$ is negligible in P, therefore certainly in B. This shows that

$$(f_1(\xi)-f(\xi))-(g_1(\eta)-g(\eta))$$

is negligible in B, so that $f_1(\xi)-g_1(\eta)$ is negligible in B if and only if $f(\xi)-g(\eta)$ is negligible in B.

It is clear that the notion of equality is reflexive, symmetric and transitive.

That the principle of permanence formulated in the preceding section is valid for this notion of equality is easy to see. Indeed, let $\mathfrak{B}^*$ be a new basis which contains all the neutrices belonging to the original basis $\mathfrak{B}$. If B^* is the sum of the neutrices belonging to the new basis, then each function negligible in B is certainly negligible in B^*. Consequently if the relation $d=d_1$ holds for the original basis $\mathfrak{B}$, it holds certainly for the new basis $\mathfrak{B}^*$.

It is not true that the inequality $d \neq d_1$ valid for the old basis $\mathfrak{B}$ is necessarily valid for the new basis $\mathfrak{B}^*$. Consequently the identities remain valid if we extend the basis but that is not necessarily the case with inequalities.

Example 10: Two distributions which have a function in common are equal, since they have the same generating function. On the other hand it may happen that two equal distributions have no function in common. Indeed that two distributions are equal means only that they contain two

functions whose difference is negligible in B, but this does not imply that this difference is identically equal to zero.

If a distribution d is generated by a constant γ and a distribution d_1 is generated by a constant γ_1, then $d = d_1$ if and only if $\gamma = \gamma_1$. Indeed that $\gamma = \gamma_1$ implies $d = d_1$ follows from the precedent example. Conversely, if $N = P$ then, according to the definition of equality, the constant $\gamma - \gamma_1$ is negligible in B, so that $\gamma - \gamma_1 = 0$

In this way we see that there is a $(1, 1)$—correspondence between the elements γ of the given range and the distributions generated by γ. We may therefore identify the distribution generated by a constant with the constant itself. Consequently the set formed by the distributions is an extension of the given range.

Section 4: Comparison between the two definitions of equality

In my previous papers on neutrices I have already given a definition of equality for distributions, but there I restricted myself always to neutrices which can be written as finite sums of neutrices with independent variables. The more general definition of equality for distributions given in the preceding section is only allowed if in the said special case the original definition agrees with the definition formulated in the preceding section, provided that we use in the basis $\mathfrak{B}$ only neutrices which can be written as finite sums of neutrices with independent variables.

To show this I assume that N is the sum $N_1 + N_2 + \dots + N_s$ of s neutrices $N_1, \dots, N_s$ with independent variables $\xi_1, \dots, \xi_s$ and that P is the sum $P_1 + P_2 + \dots + P_t$ of t neutrices $P_1, \dots, P_t$ with independent variables $\eta_1, \dots, \eta_t$. Moreover I assume for each positive integer $\sigma \leq s$ and each positive integer $\tau \leq t$ that either $\xi_\sigma = \eta_\tau$ or ξ_σ and η_τ are independent variables. In the case $\xi_\sigma = \eta_\tau$ the neutrices N_σ and P_τ have of course the same domain.

Let us denote by $\alpha_1, \dots, \alpha_r$ the variables which occur in both systems $(\xi_1, \dots, \xi_s)$ and $(\eta_1, \dots, \eta_t)$. Then $0 \leq r \leq \min(s, t)$. I shall prove:

If each neutrix belonging to the basis $\mathfrak{B}$ can be written as a finite sum of neutrices with independent variables, then a distribution d with neutrix N generated by $f(\xi_1, \dots, \xi_s)$ is, according to the definition formulated

in the preceding section, equal to a distribution d_1, with neutrix P generated by $g(\eta_1, \dots, \eta_t)$ if and only if the two distributions with neutrix B generated respectively by f and g contain a common function which depends only on $\alpha_1, \dots, \alpha_r$.

This definition of equality for distributions agrees with the original definition given in my previous papers on neutrix calculus.

Proof: That the said condition is sufficient follows from Example 10, so that we have only to show that the condition is necessary.

Let $f(\xi_1, \dots, \xi_s)$ and $g(\eta_1, \dots, \eta_t)$ be generating functions respectively of d and d_1. Assume that $d = d_1$ according to the definition given in the preceding section. Then it is possible to find a neutrix $R = \sum_{\varkappa=1}^{k} R_\varkappa$, where $R_1, \dots, R_k$ denote neutrices belonging to $\mathfrak{B}$ with independent variables $\zeta_1, \dots, \zeta_k$ such that

$$f(\xi_1, \dots, \xi_s) - g(\eta_1, \dots, \eta_t) = \sum_{\varkappa=1}^{k} \rho_\varkappa(\zeta_\varkappa), \tag{3}$$

where $\rho_\varkappa(\zeta_\varkappa)$ is negligible in $R_\varkappa$. I write $\sum_{\varkappa=1}^{k} = \sum_1 + \sum_2 + \sum_3$; here Σ_1 is extended over the positive integers $\varkappa \leqq k$ such that $\zeta_\varkappa$ occurs in the system $(\xi_1, \dots, \xi_s)$; furthermore Σ_2 is extended over the positive integers $\varkappa \leqq k$ such that $\zeta_\varkappa$ occurs in the system $(\eta_1, \dots, \eta_t)$ and not in the system $(\xi_1, \dots, \xi_s)$; finally Σ_3 is extended over the positive integers $\varkappa \leqq k$ such that $\zeta_\varkappa$ does not occur in the system $(\xi_1, \dots, \xi_s, \eta_1, \dots, \eta_t)$.

The sum Σ_3 is equal to zero, for if there exists a positive integer $\varkappa \leqq k$ which gives to Σ_3 a contribution $\rho_\varkappa(\zeta_\varkappa)$, then this contribution is according to (3) independent of $\zeta_\varkappa$ and is therefore equal to zero according to the neutrix condition of $R_\varkappa$. In this way we find by means of (3)

$$f(\xi_1, \dots, \xi_s) - \Sigma_1 = g(\eta_1, \dots, \eta_t) + \Sigma_2. \tag{4}$$

The left-hand side is independent of the variables $\eta_1, \dots, \eta_t$ which do not occur in the system $(\xi_1, \dots, \xi_s)$; the right-hand side is independent of the variables $\xi_1, \dots, \xi_s$ which do not occur in the system $(\eta_1, \dots, \eta_t)$. Consequently both sides are equal to a function $\varphi(\alpha_1, \dots, \alpha_r)$ of only the r variables $\alpha_1, \dots, \alpha_r$ which occur in both systems $(\xi_1, \dots, \xi_s)$ and $(\eta_1, \dots, \eta_t)$. This completes the proof, since the function $\varphi(\alpha_1, \dots, \alpha_r)$ occurs according to (4) in the two distributions with neutrix B generated respectively by f and g.

Section 5: Sum and difference of two distributions

If N and P are neutrices each of which can be written as a finite sum of neutrices belonging to the given basis, if d is a distribution with neutrix N generated by a function $f(\xi)$ and if d_1 is a distribution with neutrix P generated by a function $g(\eta)$, then the sum $d+d_1$ is by definition the distribution with neutrix $N+P$ generated by $f(\xi)+g(\eta)$.

This definition needs a justification. If we consider d as a distribution with neutrix N^* generated by a function $f^*(\xi^*)$ and if we consider d_1 as a distribution with neutrix P^* generated by a function $g^*(\eta^*)$, then we would find as sum of d and d_1 the distribution s^* with neutrix N^*+P^* generated by $f^*(\xi^*)+g^*(\eta^*)$. The definition of a sum given above is only allowed if the two distributions s and s^* obtained in this way are the same. Of course we assume that also N^* and P^* are finite sums of neutrices belonging to the given basis. That $s=s^*$ is obvious, since $f(\xi)-f^*(\xi^*)$ is negligible in the sum B of all the neutrices occurring in the basis. Similarly $g(\eta)-g^*(\eta^*)$ is negligible in B, so that $(f(\xi)+g(\eta))-(f^*(\xi)+g^*(\eta))$ is negligible in B, hence $s=s^*$.

From the definition given above it follows that the sum $d+d_1$ is uniquely defined, if the distributions d and d_1 and the basis are given. The addition is commutative and also associative; indeed if d, d_1 and d_2 are distributions with the compatible neutrices N, P and R, generated respectively by $f(\xi)$, , $g(\eta)$ and $h(\zeta)$, then

$$(d+d_1)+d_2=d+(d_1+d_2)$$

is the distribution with neutrix $N+P+R$, generated by $f(\xi)+g(\eta)+h(\zeta)$.

The difference $d-d_1$ is the distribution with neutrix $N+P$ generated by $f(\xi)-g(\eta)$. In this way we see that the distributions form a commutative additive group which is, according to the final remark of Section 3, an extension of the given range.

Section 6: The product of two distributions

On the range occurring in our investigation I have up until now only imposed the condition that it is an additive group, but in this section I use the stronger condition that it is a ring, so that in the range not only the addition and the subtraction but also the multiplication is possible. The

addition is commutative, but the multiplication is not necessarily commutative. The question arises whether under these circumstances the product of two distributions can be defined in such a way that for a given basis the product is uniquely defined by its two factors. This is only possible under certain supplementary conditions. To formulate these conditions I introduce three bases $\mathfrak{B}$, $\mathfrak{B}^*$ and $\mathfrak{B}^{**}$. I denote by B, B^* respectively B^{**} the sum of all the neutrices occurring in $\mathfrak{B}$, $\mathfrak{B}^*$ respectively $\mathfrak{B}^{**}$.

Let N^* and N^{**} be distributions with variables ξ^* and ξ^{**} such that N^* can be written as a finite sum of neutrices occurring in the basis $\mathfrak{B}^*$ and that N^{**} can be written as a finite sum of neutrices occurring in the basis $\mathfrak{B}^{**}$. Consider a distribution d^* with neutrix N^* generated by a function $f^*(\xi^*)$ and moreover a distribution d^{**} with neutrix N^{**} generated by a function $f^{**}(\xi^{**})$.

I need the following

Condition: The product of a function negligible in B^* with a function negligible in B^{**} is negligible in B. The product of a function negligible in B^* with $f^{**}(\xi^{**})$ is negligible in B. Finally the product of $f^*(\xi^*)$ with a function negligible in B^{**} is negligible in B.

This condition is independent of the choice of the generating functions, for if $g^*(\xi^*)$ and $g^{**}(\xi^{**})$ are arbitrary generating functions of d^* and d^{**}, then $g^*(\xi^*) - f^*(\xi^*)$ and $g^{**}(\xi^{**}) - f^{**}(\xi^{**})$ are negligible respectively in B^* and B^{**}, so that the condition remains true if we replace $f^*(\xi^*)$ and $f^{**}(\xi^{**})$ by $g^*(\xi^*)$ and $g^{**}(\xi^{**})$.

If the said condition holds, then for each function $\nu^*(\xi^*)$ negligible in N^* and for each function $\nu^{**}(\xi^{**})$ negligible in N^{**} the function

$$(f^*(\xi^*) + \nu^*(\xi^*))(f^{**}(\xi^{**}) + \nu^{**}(\xi^{**})) - f^*(\xi^*)\, f^{**}(\xi^{**})$$

is the sum of three terms each negligible in B. Consequently the product

$$(f^*(\xi^*) + \nu^*(\xi^*))(f^{**}(\xi^{**}) + \nu^{**}(\xi^{**}))$$

generates with B the same distribution as $f^*(\xi^*)\, f^{**}(\xi^{**})$. This neutrix which is for given bases $\mathfrak{B}$, $\mathfrak{B}^*$ and $\mathfrak{B}^{**}$ uniquely defined by the distributions d^* and d^{**} is called the product $d^*\, d^{**}$.

The definition given above yields the distributive law: if $d^*\, d^{**}$ and $d_1^*\, d^{**}$ exist, then

$$d^* d^{**} + d_1^* d^{**} = (d^* + d_1^*) d^{**};$$

if $d^* d^{**}$ and $d^* d_1^{**}$ exist, then

$$d^* d^{**} + d^* d_1^{**} = d^* (d^{**} + d_1^{**}).$$

Going on in this way we can define the product of more than two distributions with the distributive and associative laws.

Section 7: Series

The purpose of this section is to introduce series of the from $\sum_{h=0}^{\infty} d_h$. To this end I need the notion of type. A type is a class formed by convergent series $\sum_{h=0}^{\infty} a_h$ such that any two series $\sum_{h=0}^{\infty} a_h$ and $\sum_{h=0}^{\infty} b_h$ belonging to it have the property that also the series $\sum_{h=0}^{\infty} (a_h - b_h)$ belongs to the class. Then this class contains each of the series

$$\sum_{h=0}^{\infty} (a_h - a_h) = \sum_{h=0}^{\infty} 0; \quad \sum_{h=0}^{\infty} (0 - b_h) = \sum_{h=0}^{\infty} (-b_h);$$

$$\sum_{h=0}^{\infty} (a_h - (-b_h)) = \sum_{h=0}^{\infty} (a_h + b_h),$$

so that not only subtraction but also addition is allowed.

In analysis we often restrict ourselves to a particular type, for instance to the type formed by the absolutely convergent series, or to the type formed by the uniformly convergent series, or to the type formed by the series which converge faster than any geometric series with ratio $\neq 0$, or to the type formed by the series $\sum_{h=0}^{\infty} a_h$ with $a_h = O(h^{-2})$, and so on.

Let us now introduce a series $d_0 + d_1 + \ldots$, where d_h is a distribution with neutrix N_h and where each $N_h (h = 0, 1, \ldots)$ can be written as a finite sum of neutrices occurring in the basis $\mathfrak{B}$. Denote by B the sum of all the neutrices occurring in this basis; let ξ and B' denote the variable and the domain of B. Each function negligible in N_h is negligible in B, so that each distribution d_h occurring in the series $d_0 + d_1 + \ldots$ can be

considered as a distribution with neutrix B. I do this for the sake of simplicity. I restrict myself to types with the following property: if $\beta_h(\xi)\,(h = 0, 1, \ldots)$ denote functions negligible in B such that for each element ξ of B' the series $\sum_{h=0}^{\infty} \beta_h(\xi)$ belongs to the given type, then the sum of the series is negligible in B. If it is possible to find in $d_h\,(h = 0, 1, \ldots)$ a function $g_h(\xi)$ such that the series $\sum_{h=0}^{\infty} g_h(\xi)$ belongs to the given type, then the distribution d with neutrix B generated by $\sum_{h=0}^{\infty} g_h(\xi)$ is independent of the choice of the functions $g_h(\xi)\,(h = 0, 1, \ldots)$. Indeed, if $d_h\,(h = 0, 1, \ldots)$ contains a function $g_h^*(\xi)$ such that the series $\sum_{h=0}^{\infty} g_h^*(\xi)$ belongs to the given type, then

$$\sum_{h=0}^{\infty} (g_h^*(\xi) - g_h(\xi))$$

is a series of the given type in which each term is negligible in B, so that its sum is negligible in B; this implies that the two functions $\sum_{h=0}^{\infty} g_h(\xi)$ and $\sum_{h=0}^{\infty} g_h^*(\xi)$ generate with B the same distribution d. In this case I call the series $d_0 + d_1 + \ldots$ convergent with sum d.

Example 11: Let B be the neutrix with domain $\xi > 0$ and with as negligible functions those functions of ξ which tend for $\xi \to \infty$ to zero. The type formed by the uniformly convergent series satisfies the condition formulated above. Consequently if each distribution $d_h\,(h = 0, 1, \ldots)$ contains a function $g_h(\xi)$ such that $\sum_{h=0}^{\infty} g_h(\xi)$ converges uniformly, then $d_0 + d_1 + \ldots$ is the distribution formed by the functions

$$\sum_{h=0}^{\infty} g_h(\xi) + o(1).$$

Example 12: Let p be a real number. Let $\mathfrak{A}$ be an additive group formed by functions $f(\xi)$ defined for $\xi > p$ such that each function belonging

to $\mathfrak{A}$ which is not identically equal to zero is unbounded as $\xi \to \infty$. Let B be the neutrix with domain $\xi > p$ formed by the functions of the form $f(\xi) + o(1)$, where $f(\xi)$ denotes an arbitrary function belonging to $\mathfrak{A}$ and where $o(1)$ denotes a function of ξ which tends to zero as $\xi \to \infty$. If $d_h (h = 0, 1, \ldots)$ is a distribution with neutrix B generated by a constant a_h such that $\sum_{h=0}^{\infty} a_h$ converges, then

$$\sum_{h=0}^{\infty} d_h = \sum_{h=0}^{\infty} a_h, \tag{5}$$

if we restrict ourselves to uniformly convergent series.

Proof: B satisfies the neutrix condition, for if $f(\xi) + o(1) = \gamma$, where γ is independent of ξ, then $f(\xi)$ is bounded and therefore equal to zero, so that $\gamma = 0$.

We must prove that any uniformly convergent series

$$\sum_{h=0}^{\infty} \psi_h(\xi) \tag{6}$$

in which each term is negligible in B has a sum which is negligible in B. Then

$$\psi_h(\xi) = f_h(\xi) + \varepsilon_h(\xi)$$

where $f_h(\xi)$ belongs to $\mathfrak{A}$ and where $\varepsilon_h(\xi) \to 0$ as $\xi \to \infty$. According to the uniform convergence of (6) there exists an integer $q \geqq 0$ such that for each choice of the integers s and t with $t \geqq s \geqq q$

$$\left| \sum_{h=s}^{t} (f_h(\xi) + \varepsilon_h(\xi)) \right| < 1,$$

hence for sufficiently large ξ

$$\left| \sum_{h=s}^{t} f_h(\xi) \right| < 2.$$

The function $\sum_{h=s}^{t} f_h(\xi)$ is bounded as $\xi \to \infty$. It belongs to $\mathfrak{A}$ and is therefore identically equal to zero. This holds for each choice of the integers

s and t with $t \geqq s \geqq q$, so that $f_h(\xi) = 0$ for $h \geqq q$. Consequently

$$\sum_{h=0}^{\infty} (f_h(\xi) + \varepsilon_h(\xi)) = \sum_{h=0}^{q-1} f_h(\xi) + \sum_{h=0}^{\infty} \varepsilon_h(\xi). \tag{7}$$

The sum $\sum_{h=0}^{q-1}$ represents a function occurring in $\mathfrak{A}$ and the last term in (7) denotes a uniformly convergent series in which each term tends for $\xi \to \infty$ to zero, so that also the last term in (7) tends to zero. Consequently the right-hand side of (7) represents a function negligible in B, so that the type formed by the uniformly convergent series satisfies the prescribed condition.

Since the right-hand side of (5) represents a convergent series independent of ξ, by definition $\sum_{h=0}^{\infty} d_h$ is the distribution with neutrix N generated by $\sum_{0}^{\infty} a_h$. This distribution is generated by a constant and therefore equal to this constant. This completes the proof.

Section 8: Linear operators

Denote the range by $\mathfrak{A}$. Let λ be a linear operator which transforms some elements α of $\mathfrak{A}$ into uniquely determined elements $\lambda\alpha$ of $\mathfrak{A}$. That the operator λ is linear means: if two elements α and β of $\mathfrak{A}$ have the images $\lambda\alpha$ and $\lambda\beta$, then $\alpha + \beta$ and $\alpha - \beta$ have the images $\lambda\alpha + \lambda\beta$ and $\lambda\alpha - \lambda\beta$.

Let $f(\xi)$ be a function defined for each element ξ of a certain domain N'. I say that the linear operator λ can be applied on the function $f(\xi)$ if for each element ξ of N' $f(\xi)$ and $\lambda f(\xi)$ denote elements of the range $\mathfrak{A}$. Let N be a neutrix with variable ξ and domain N'. I say that the linear operator λ can be applied on the neutrix N if it is impossible to find a function $\nu(\xi)$ negligible in N on which the operator λ can be applied with the property that $\lambda\nu(\xi)$ is for each element ξ of N' equal to a constant $\neq 0$. Then the functions $\lambda\nu(\xi)$, where $\nu(\xi)$ denotes an arbitrary function negligible in N on which the operator λ can be applied, form a neutrix.

I denote this neutrix by λN. The neutrices N and λN are not necessarily compatible.

Let B denote the sum of all the neutrices occurring in the basis and let λ be a linear operator with the following property: if λ can be applied on a function $\beta(\zeta)$ negligible in B, then $\lambda\beta(\zeta)$ is negligible in B. Then it is impossible to find a function $\beta(\zeta)$ negligible in B on which the operator λ can be applied in such a way that $\lambda\beta(\zeta)$ is for all elements ζ of B' equal to a constant $\neq 0$. Consequently the neutrix λB exists and B is an enlargement of λB.

Let d be a distribution with neutrix N, where N is a neutrix with variable ξ which can be written as a finite sum of neutrices occurring in the basis. If it is possible to find in d a function $g(\xi)$ on which the operator λ can be applied, then the distribution with neutrix B generated by $\lambda g(\xi)$ is uniquely determined if the basis, the distribution d and the operator λ are given. Indeed, if d is considered as a distribution with a neutrix N^* such that it contains a function $g^*(\xi^*)$ on which the operator λ can be applied, then $g^*(\xi^*) - g(\xi)$ is negligible in B, so that also $\lambda g^*(\xi) - \lambda g(\xi)$ is negligible in B; consequently $\lambda g(\xi)$ and $\lambda g^*(\xi^*)$ generate with the neutrix B the same distribution. This distribution is denoted by λd and is called the λ-transform of d. If the distribution d and the operator λ are given and if the basis is known, then the λ-transform of d is uniquely determined.

Let μ be a linear operator which satisfies the conditions formulated above, namely: if μ can be applied on a function $\beta(\zeta)$ negligible in B, then $\mu\beta(\zeta)$ is negligible in B. If the distribution λd contains a function on which the operator μ can be applied, then $\mu\lambda d$ exists. If d contains a function $f(\xi)$ for which

$$\mu\lambda f(\xi) = \lambda\mu f(\xi),$$

then $\mu\lambda d = \lambda\mu d$. This is an example of the general rule that in the theory of the linear operators the properties of a distribution are determined, not by the pathological but by the simple functions occurring in the distribution.

Section 9: Introduction of a supplementary variable

By definition a neutrix N is an additive group formed by functions $\nu(\xi)$ of a variable ξ which traverses the domain N' of N with the property

that each function $\nu(\xi)$, which is for all elements ξ of N' independent of ξ, is identically equal to zero. Often it is useful to introduce a supplementary variable x which traverses a given domain $\mathfrak{X}$. If we do this we change the definition of a neutrix as follows.

A neutrix N is an additive group formed by functions $\nu(\xi, x)$ of a variable ξ which traverses the domain N' of N and a variable x which traverses a given domain $\mathfrak{X}$ with the property that each function $\nu(\xi, x)$ which is for all elements ξ of N' and all elements x of $\mathfrak{X}$ independent of ξ, is identically equal to zero.

In this case I shall call ξ the variable of N and I call x the supplementary variable.

In order to apply the theory of linear operators on neutrices N of this kind, I assume that the range $\mathfrak{A}$ is formed by functions $\alpha(x)$ of x defined for each element x of $\mathfrak{X}$. Moreover I introduce a linear operator λ which can be applied on some functions $\alpha(x)$ belonging to the range. Let $f(\xi, x)$ be a function of ξ and x which denotes for each choice of the element ξ of N' a function of x which occurs in $\mathfrak{A}$. I say that the operator λ can be applied on $f(\xi, x)$ if for each ξ in N' also $\lambda f(\xi, x)$ denotes a function of x which occurs in $\mathfrak{A}$. Let N be a neutrix with variable ξ and with supplementary variable x. I say that the linear operator λ can be applied on N if each function $\nu(\xi, x)$ negligible in N on which the operator λ can be applied with the property that $\lambda\nu(\xi, x)$ is independent of ξ satisfies the condition that $\lambda\nu(\xi, x)$ is identically equal to zero. In this case the functions $\lambda\nu(\xi, x)$, where $\nu(\xi, x)$ denotes an arbitrary function negligible in N on which the operator λ can be applied, form a neutrix with variable ξ and supplementary variable x. I denote this neutrix by λN. In the special case that each function $\alpha(x)$ occurring in $\mathfrak{A}$ on which the operator λ can be applied has the property that $\lambda\alpha(x)$ is independent of x, then the supplementary variable x drops out in the neutrix λN.

Let B denote the sum of all the neutrices occurring in the basis; the neutrices occurring in the basis may involve the supplementary variable x. Let λ be a linear operator with the following property: if λ can be applied on a function $\beta(\zeta, x)$ negligible in B, then $\lambda\beta(\zeta, x)$ is negligible in B. In this case λB exists and B is an enlargement of λB.

Let N be a neutrix with variable ξ and supplementary variable x which can be written as a finite sum of neutrices occurring in the basis. If it is possible to find in a distribution d with neutrix N a function $g(\xi, x)$ on which the operator λ can be applied, then λd is the distribution with neutrix B generated by $\lambda g(\xi, x)$.

In the Sections 10, 11 and 12 I treat special linear operators.

Section 10: Limits

I apply the theory of linear operators developed in the preceding section in the special case that for each function $\alpha(x)$ occurring in the range $\mathfrak{A}$ on which the operator λ can be applied $\lambda\alpha(x)$ is independent of x. In that case I call $\lambda\alpha(x)$ the limit of $\alpha(x)$ and I write

$$\lambda\alpha(x) = \lim_{x} \alpha(x).$$

In this way we obtain the following results:

Let N be a neutrix with variable ξ and supplementary variable x such that each function $\nu(\xi, x)$ negligible in N for which $\lim\limits_{x} \nu(\xi, x)$ exists and is independent of ξ has the property that this limit is equal to zero. Then $\lim\limits_{x} N$ is the neutrix formed by $\lim\limits_{x} \nu(\xi, x)$, where $\nu(\xi, x)$ denotes an arbitrary function negligible in N for which the limit exists.

If it is possible to find in a distribution d with neutrix N a function $g(\xi, x)$ for which $\lim\limits_{x} g(\xi, x)$ exists, then $\lim\limits_{x} d$ is the distribution with neutrix B generated by $\lim\limits_{x} g(\xi, x)$.

The theory of convergent series given in Section 7 can be treated as a particular case of the theory of limits if we use for $\mathfrak{X}$ the set formed by the integers $x \geqq 0$ and if we say that on $\sum\limits_{h=0}^{x} a_h$ the operator λ with

$$\lambda \sum_{h=0}^{x} a_h = \lim_{x} \sum_{h=0}^{x} a_h$$

can be applied if the series $\sum\limits_{h=0}^{\infty} a_h$ belongs to the given type.

Section 11: Differentiation

I apply the theory of linear operators developed in Section 9 with the understanding that I call $\lambda\alpha(x)$, if it exists, the derivative of $\alpha(x)$ with respect to x.

In this way we obtain the following results:

Let N be a neutrix with variable ξ and supplementary variable x such that each function $\nu(\xi, x)$ negligible in N of which the partial derivative with respect to x exists and is independent of ξ satisfies the condition that this partial derivative is identically equal to zero. Then the derivative with respect to x of N is the neutrix formed by the functions $\frac{\partial \nu(\xi, x)}{\partial x}$, where $\nu(\xi, x)$ denotes an arbitrary function negligible in N which is partially differentiable with respect to x.

If it is possible to find in a distribution d with neutrix N a function $g(\xi, x)$ which is partially differentiable with respect to x, then the derivative of d with respect to x is the distribution with neutrix B generated by $\frac{\partial g}{\partial x}$.

Section 12: Transforms

To show that the theory of the linear operators contains the theory of the transforms, I treat here the Fourier transforms. Let N be a neutrix with variable ξ, with supplementary variable x and with domain N'. I say that the operator λ can be applied on a function $f(\xi, x)$ defined for each element ξ of N' and each real x, if the Fourier transform

$$\lambda f(\xi, x) = \int_{-\infty}^{\infty} f(\xi, t)\, e^{-2\pi i x t}\, dt$$

exists for each ξ in N' and each real x. Assume that N satisfies the condition that each function $\nu(\xi, x)$ negligible in N the Fourier transform of which exists and is independent of ξ has the property that this Fourier transform is identically equal to zero. Then the Fourier transform of N is the neutrix formed by all the Fourier transforms of the negligible functions of N, in as far as these transforms exist.

If it is possible to find in a distribution d with neutrix N a function

$g(\xi, x)$ the Fourier transform of which exists, then the Fourier transform of d is the distribution with neutrix B generated by the Fourier transform of $g(\xi, x)$.

Section 13: Enlargement of the basis by means of an operator

Let N be a neutrix which can be written as a finite sum of neutrices occurring in the basis. Let λ be a linear operator which can be applied on N. If each function negligible in λN is negligible in B, then we can apply the theory exposed in the preceding sections. Suppose that this is not the case but that on the other hand λN is compatible with all the neutrices occurring in the basis $\mathfrak{B}$. Then we can introduce the basis $\mathfrak{B}^*$ formed by the neutrix λN and by all the neutrices belonging to $\mathfrak{B}$. Each function negligible in λN is certainly negligible in the sum B^* of the neutrices belonging to $\mathfrak{B}^*$, so that we can apply the theory developed in this paper with $\mathfrak{B}$ replaced by $\mathfrak{B}^*$ and with B replaced by B^*.

So we can go on as far as at each step we have to deal with a neutrix which is compatible with all the neutrices which at that moment belong to the basis, but it may be that each application of a linear operator requires an enlargement of the basis.

As an example I consider the neutrix N_0 whose domain is formed by the positive integers ξ and where a function $v(\xi, x)$ ($\xi = 1, 2, \ldots$; $0 \leqq x \leqq 1$) continuous in x is negligible if and only if for each fixed integer $n \geqq 0$

$$\lim_{\xi \to \infty} \int_0^1 x^n v(\xi, x)\, dx = 0. \tag{8}$$

N_0 is a neutrix according to the following lemma of Lerch: a continuous function $v(x)$ ($0 \leqq x \leqq 1$) which satisfies for each integer $n \geqq 0$ the relation

$$\int_0^1 x^n v(x)\, dx = 0$$

is identically equal to zero. This lemma is obvious. Indeed it is possible to find polynomials $p_h(x)$ ($h = 0, 1, \ldots$) which tend for $h \to \infty$ in the

interval $0 \leqq x \leqq 1$ to $\nu(x)$, uniformly in x; if $\overline{p_h}(x)$ denotes the complex conjugated polynomial of $p_h(x)$, then we have for $h \to \infty$

$$\int_0^1 x^n |\nu(x)|^2 \, dx = \int_0^1 x^n (\nu(x) - p_h(x)) \, \nu(x) \, dx \to 0,$$

hence $\nu(x) = 0$.

I say that the operator λ can be applied on a function $f(\xi, x)$ of ξ and x if for each positive integer ξ and for $0 \leqq x \leqq 1$ the partial derivative of $f(\xi, x)$ with respect to x exists and is a continuous function of x; this partial derivative is then denoted by $\lambda f(\xi, x)$. We shall prove that λN_0 exists and is compatible with N_0. That N_0 is not an enlargement of λN_0 follows from the fact that the function

$$\nu(\xi, x) = \begin{cases} 0 & \text{for } 0 \leqq x \leqq \dfrac{\xi}{\xi+1} \\[2ex] ((\xi+1)x - \xi)^2 & \text{for } \dfrac{\xi}{\xi+1} \leqq x \leqq 1 \end{cases}$$

is negligible in N_0 and has a continuous derivative with respect to x which is not negligible in N_0, since

$$\int_0^1 \frac{\partial \nu(\xi; x)}{\partial x} \, dx = \nu(\xi, 1) - \nu(\xi, 0) = 1 \neq 0.$$

Consider the case that the basis $\mathfrak{B}_0$ is formed by the neutrix N_0. By adjoining λN_0 we obtain a basis $\mathfrak{B}_1$ formed by the two neutrices N_0 and λN_0. Let B_1 be the sum of these two neutrices, so that B_1 is an enlargement of λN_0 and the method exposed in this paper can be applied with the following result: if we use the basis $\mathfrak{B}_0$, then the neutrix N_0 is not continuously differentiable with respect to x. However, if we apply the basis $\mathfrak{B}_1$, then N_0 is continuously differentiable with respect to x and each distribution with neutrix N_0 which contains at least one function which is continuously differentiable with respect to x is itself continuously differentiable with respect to x.

We can go on. We can namely show that all the neutrices $N_0, \lambda N_0, \lambda^2 N_0, \ldots$ exist and are compatible. If the basis $\mathfrak{B}$ is formed by

these neutrices, then $B = \sum_{h=0}^{\infty} \lambda^h N_0$ is an enlargement of each $\lambda^h N_0$. Consequently if we use this basis $\mathfrak{B}$, then N_0 is infinitely often differentiable with respect to x and each distribution with neutrix N_0, which contains at least one function $f(\xi, x)$ which is for each positive integer ξ infinitely often differentiable with respect to x $(0 \leqq x \leqq 1)$, is itself infinitely often differentiable with respect to x. For instance each function $f(x)$ which is independent of ξ and is continuous in x in the interval $0 \leqq x \leqq 1$, generates with N_0 a distribution which is infinitely often differentiable with respect to x. Indeed, according to a well known approximation theorem it is possible to find a sequence formed by functions $f(\xi, x)$ $(\xi = 1, 2, \ldots)$ which are infinitely often differentiable with respect to x such that $f(\xi, x)$ tends, uniformly in x, to $f(x)$ as $\xi \to \infty$; consequently the distribution contains the infinitely often differentiable function $f(\xi, x)$.

Finally we must show that the neutrices N_0, λN_0, $\lambda^2 N_0$, ... exist and are compatible. Let k be an integer $\geqq 0$. We assume that we have proved already that N_0, λN_0, ..., $\lambda^k N_0$ exist and are compatible, so that $N_k = \sum_{h=0}^{k} \lambda^h N_0$ exists and we assume moreover that each function $\mu(\xi, x)$ negligible in N_k has the property that for each integer $n \geqq 0$

$$(9) \qquad \int_0^1 x^n \mu(\xi, x)\, dx = \sum_{\substack{h=0 \\ h \leqq n}}^{k} p_h(n) \int_0^1 x^{n-h} \nu_h(\xi, x)\, dx + q(\xi, n),$$

where $p_h(n)$ is a polynomial in n, where $q(\xi, n)$ is a polynomial in n whose coefficients may depend on ξ and where $\nu_h(\xi, x)$ is negligible in N_0.

It is obvious that this is true for $k = 0$ and I shall show that it remains true if k is replaced by $k+1$. In the construction of $\lambda^{k+1} N_0$ we consider a function $\mu(\xi, x)$ negligible in $\lambda^k N_0$ which is continuously differentiable with respect to x.

For each integer $n \geqq 0$ we have

$$(10) \qquad \int_0^1 x^n \frac{\partial \mu}{\partial x}\, dx = x^n \mu(\xi, x) \Big|_0^1 - n \int_0^1 x^{n-1} \mu(\xi, x)\, dx.$$

Consequently each function

$$\sigma(\xi, x) = \rho(\xi, x) + \frac{\partial \mu(\xi, x)}{\partial x},$$

where $\rho(\xi, x)$ is negligible in N_k and where $\mu(\xi, x)$, negligible in $\lambda^k N_0$, is continuously differentiable with respect to x, has the property that

$$\int_0^1 x^n \sigma(\xi, x)\, dx = \sum_{\substack{h=0 \\ h \leq n}}^{k+1} p_h^*(n) \int_0^1 x^{n-h} \nu_h^*(\xi, x)\, dx + q^*(\xi, n), \tag{11}$$

where $p_h^*(n)$ and $q^*(\xi, n)$ are polynomials in n and where $\nu_h^*(\xi, x)$ is negligible in N_0. To prove that $N_0, \lambda N_0, \ldots, \lambda^{k+1} N_0$ are compatible it is sufficient to show that each continuous function $\sigma(\xi, x)$ with (11) which is independent of ξ is identically equal to zero. Put $\sigma(\xi, x) = \sigma(x)$. According to the definition of the neutrix N_0 each term of the sum Σ occurring in (11) tends for $\xi \to \infty$ to zero, so that for each fixed integer $n \geqq 0$

$$\lim_{\xi \to \infty} q^*(\xi, n) = \int_0^1 x^n \sigma(x)\, dx. \tag{12}$$

Consequently each term in the polynomial $q^*(\xi, n)$ tends for $\xi \to \infty$ to a finite limit. This limit is equal to zero since the right hand side of (12) tends for $n \to \infty$ to zero. In this way we see that the right hand side of (12) is equal to zero for each integer $n \geqq 0$, so that $\sigma(x)$ is identically equal to zero according to the lemma of Lerch. This completes the proof.

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THE DEPENDENCE OF THE SCHOENFLIES EXTENSION ON AN ACCESSORY PARAMETER

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Part I. REDUCTION OF THE PROBLEM

§ 1. Introduction.

We begin with a definition of the Schoenflies extension problem. Cf. Refs. 1 to 5 and 13. The problem without a parameter, as defined and solved in Ref. 5, will be called the simple problem, as distinguished from the problem with parameter.

Let E be an euclidean n-space, $n>1$, of points $\mathbf{x}$, with rectangular coordinates $(x_1, \ldots, x_n)$. Let S be an $(n-1)$-sphere of unit radius with center Z. In Part I we suppose that Z is the origin. If Σ is a topological $(n-1)$-sphere in E let $J\Sigma$ denote the closure of the interior of Σ, and $\mathring{J}\Sigma$ the interior of Σ.

A C^m-diffeomorphism, $m>0$, of an open subset A of E into E will have its usual meaning when $m>0$ (Ref. 3, § 0), and when $m=0$ shall be merely a homeomorphism of A into E (Ref. 4).

Let N be an open neighborhood of S relative to $E-Z$. Let φ be a C^m-diffeomorphism of N into E which maps points of N interior (exterior) to S into points of E interior (exterior) to $\varphi(S)$. As a matter of permanent notation, set

$$JS - Z = J_0 S, \quad N_e = N - JS.$$

The following theorem is a consequence of Theorem 1.2 of Ref. 14.

Theorem 1.1. *If $N_* \subset N$ is a sufficiently small open neighborhood of S there exists a homeomorphism Λ_φ of $N \cup JS$ into E which is an extension of $\varphi | (N_e \cup N_*)$ and which, when $m>0$, is, in addition, a C^m-diffeomorphism of $N \cup J_0 S$ into E. As a consequence*

$$\Lambda_\varphi(JS) = J\varphi(S). \tag{1.1}$$

The mapping Λ_φ may be a C^m-diffeomorphism over $\mathring{J}S$ when $m>0$, but there exist values of n, the dimension of E, and mappings φ for which the extension Λ_φ cannot be affirmed to be a C^m-diffeomorphism over $\overset{c}{J}S$. Cf. Ref. 6. The above theorem remains true if the point Z is replaced by any other fixed point Z_1 in $\mathring{J}S$, provided of course N_* is replaced by a suitably chosen neighborhood of S dependent on φ and Z_1. Our choice of Z as the possible singular point of Λ_φ, when $m>0$, is dictated by considerations of simplicity of exposition. No generality is thereby lost. See Ref. 14.

Given the elements (φ, N, m), conditioned as above, the corresponding simple Schoenflies problem is to find Λ_φ and N_* such that Theorem 1.1 holds. We set

$$(\varphi, N, m) = P \tag{1.2}$$

and refer to P as the given simple problem and to (Λ_φ, N_*) as a solution of P.

Product spaces. We shall recall some of the notation associated with product spaces. Cf. Ref. 7.

Let U and V be two spaces. Let u and v be points in U and V respectively, and let X be a subset of $U \times V$. One introduces the projections

$$\begin{cases} \mathrm{pr}_1 : U \times V \to U \ ; \ \mathrm{pr}_1(u, v) = u \\ \mathrm{pr}_2 : U \times V \to V \ ; \ \mathrm{pr}_2(u, v) = v \, . \end{cases} \tag{1.3}$$

For p fixed in V set

$$X^p = \{u \mid (u, p) \in X\} \, . \tag{1.4}$$

We shall have no need for subsets $\{v \mid (q, v) \in X\}$ for which q is fixed in U.

Let $(u, v) \to F(u, v)$ be a mapping of X into $U \times V$. Set

$$\mathrm{pr}_1 F(u, v) = F_1(u, v) \ , \ \mathrm{pr}_2 F(u, v) = F_2(u, v) \, .$$

Thus $(u, v) \to F_1(u, v)$ is a mapping of X into U, and $(u, v) \to F_2(u, v)$ is a mapping of X into V.

Definition. A mapping F of a subset X of $U \times V$ into $U \times V$ will be termed v-invariant if for each $(u, v) \in X$, $F_2(u, v) = v$. For $(u, v) \in X$ one sets

$$F_1(u, v) = F^v(u) \tag{1.5}$$

and for fixed $p \in \mathrm{pr}_2 X$ denotes the partial mapping $u \to F_1(u, p)$ of X^p into U by F^p. We term F^p the p-section of F.

Mappings which are u-invariant are not needed.

For a v-invariant mapping F of X into $U \times V$ to be biunique it is necessary and sufficient that F^p be a biunique mapping of X^p into U for each $p \in \mathrm{pr}_2 X$. If U and V are topological spaces a necessary and sufficient condition that a p-invariant mapping F of X into $U \times V$ be continuous is that F_1 be a continuous mapping of X into U.

The proof of the above statements is immediate. The proof of Lemma 1.1 below is somewhat deeper and will be given in § 2. Lemma 1.1 in the topological case, and Lemma 1.2 in the differentiable case, prepare the way for the principal theorem of this paper.

Lemma 1.1. Let Γ be a topological space of points p and X an open subset of $E \times \Gamma$. A necessary and sufficient condition that a p-invariant mapping F of X into $E \times \Gamma$ be a homeomorphic mapping is that F_1 be a continuous mapping of X into E and that the mapping F^p of X^p into E be biunique for each $p \in \mathrm{pr}_2 X$.

The Schoenflies problem with a parameter. Corresponding to each integer $m = 0, 1, \ldots$ and to $m = +\infty$ we shall introduce a space Γ_m on which will range a point p, termed a parameter in a Γ_m-Schoenflies problem. The case in which $m = 0$ will be termed the topological case, and the case in which $m > 0$ the differentiable case.

When $m = 0$, Γ_m shall be an arbitrary paracompact space. When $m > 0$, Γ_m shall be a connected differentiable r-manifold, $r > 0$, of class C^m with a countable base. Cf. Ref. 3, § 0.

In the differentiable case, $m > 0$, we regard $E \times \Gamma_m$ as a differentiable $(n+r)$-manifold, with a differential structure determined by the differential structures of E and Γ_m in the usual way. We can now state Lemma 1.2. Lemma 1.2 is proved in § 2.

Lemma 1.2. Let X be an open subset of the space

$E \times \Gamma_m$, $m > 0$. A necessary and sufficient condition that a p-invariant map

$$F : X \to E \times \Gamma_m \; ; \; (\mathbf{x}, p) \to F(\mathbf{x}, p)$$

be a C^m-diffeomorphism of X into $E \times \Gamma_m$ is that F_1 be a map of X into E of class C^m, and that for each $p \in \mathrm{pr}_2 X$, F^p be a C^m-diffeomorphism of X^p into E.

The reader should recall that X^p is open relative to E, since X is open relative to $E \times \Gamma_m$. Cf. Ref. 8, p. 65.

The Γ_m-problem. The data required to define a Schoenflies problem with a parameter space Γ_m include an open neighborhood L of $S \times \Gamma_m$ relative to $(E - Z) \times \Gamma_m$, where Z is the center of S. There is further given a p-invariant C^m-diffeomorphism

$$\Phi : L \to E \times \Gamma_m \; ; \; (\mathbf{x}, p) \to \Phi(\mathbf{x}, p) \tag{1.6}$$

such that for each $p \in \Gamma_m$, Φ^p maps points of L^p which are interior (exterior) to S into points which are interior (exterior) to the topological $(n-1)$-sphere $\Phi^p(S)$. As a matter of permanent notation set

$$L_e = L - (JS \times \Gamma_m). \tag{1.6$'$}$$

The principal theorem of this paper can now be stated as follows.

Theorem 1.2. Corresponding to (Φ, L, Γ_m), conditioned as above, and to any sufficiently small open neighborhood $L_* \subset L$ of $S \times \Gamma_m$ relative to $E \times \Gamma_m$, there exists a p-invariant homeomorphism

$$\Lambda_\Phi : L \cup (JS \times \Gamma_m) \to E \times \Gamma_m \tag{1.7}$$

which extends $\Phi | (L_e \cup L_*)$ and which, when $m > 0$, is, in addition, a C^m-diffeomorphism into $E \times \Gamma_m$ of

$$L \cup (J_0 S \times \Gamma_m). \tag{1.8}$$

Given the elements (Φ, L, Γ_m), conditioned as above, the corresponding Schoenflies Γ_m-problem is, by definition, to find Λ_Φ and L_* so that Theorem 1.2 is satisfied.

We set $\mathbf{P} = (\Phi, L, \Gamma_m)$ and refer to $\mathbf{P}$ as a Γ_m-problem and to (Λ_Φ, L_*) as a solution of $\mathbf{P}$, provided Theorem 1.2 is satisfied by (Λ_Φ, L_*).

Convention. Two Γ_m-problems

$$\mathbf{P} = (\Phi , L , \Gamma_m) \qquad \mathbf{P}' = (\Phi' , L' , \Gamma_m)$$

are said to be equal and one writes $\mathbf{P} = \mathbf{P}'$, if and only if $L = L'$, and $\Phi = \Phi'$.

Methods. A Γ_m-problem $\mathbf{P} = (\Phi , L , m)$ defines for each $p \in \Gamma_m$ a simple problem

$$\mathbf{P}^p = (\Phi^p , L^p , m) \tag{1.9}$$

termed the p-section of $\mathbf{P}$, for which a solution exists in accord with Theorem 1.1. However the solutions affirmed to exist in Theorem 1.1 are not unique, and a solution chosen arbitrarily for each p will certainly not lead to a solution of problem $\mathbf{P}$ in the sense of Theorem 1.2. Our task is then so to define a solution of each simple problem $\mathbf{P}^p$ that the resulting family of solutions will lead to a solution of $\mathbf{P}$ in the sense of Theorem 1.2.

Our method is to use properly chosen p-invariant C^m-diffeomorphisms of open subsets of $E \times \Gamma_m$ into $E \times \Gamma_m$ so as to transform a given Γ_m-problem into a simpler "equivalent" Γ_m-problem (see § 6) and in particular into a uniform Γ_m-problem. Equivalent Γ_m problems have the same parameter space Γ_m.

Uniform Γ_m-problems. An arbitrary Γ_m-problem may have the property that the set

$$\bigcap_{p \in \Gamma_m} L^p \tag{1.10}$$

is not a neighborhood of S. One of several fundamental properties of a uniform Γ_m-problem is that for such a problem the set (1.10) is a neighborhood of S.

A second fundamental property of a uniform Γ_m-problem may be described as follows. We refer to the simple problem P, given in (1.2), and let Q be the point $(0, ..., 0, 1)$ on S. In Ref. 4 Morse has shown that there exists a C^m-diffeomorphism f of E into E such that

$$(f\varphi , N , m) \tag{1.11}$$

is a simple problem equivalent (see § 3) to the given problem P, one in which $f\varphi$ reduces to the identity on a small spherical n-ball B with

center at Q. If now $\mathbf{P}$ is a uniform Γ_m-problem it is similarly possible (Cf. § 12) to define a p-invariant C^m-diffeomorphism F of $E \times \Gamma_m$ into $E \times \Gamma_m$ such that

$$(F\Phi, L, \Gamma_m) \tag{1.12}$$

is a Γ_m-problem equivalent to $\mathbf{P}$, one in which $F\Phi$ reduces to the identity on $B \times \Gamma_m$ for a suitable choice of the above ball B. Note that the choice of B is independent of $p \in \Gamma_m$.

Because of these properties of a uniform Γ_m-problem it is possible to utilize our explicit formulas for the solution of a simple problem, as presented in Ref. 5, to obtain a solution of the Γ_m-problem (1.12). A solution of (1.12) will imply a solution of the original Γ_m-problem.

§ 2. A Brouwer theorem generalized.

Before coming to the proof of Lemma 1.1 we shall recall certain results concerning essential maps of an $(n-1)$-sphere $\mathfrak{S}$ in E into $\mathfrak{S}$. Let $\mathbf{O}$ be the center of $\mathfrak{S}$. Let Σ_i, $i = 0, 1$, be two topological $(n-1)$-spheres in E which do not meet $\mathbf{O}$. We shall make the following statement precise. If $\mathring{J}\Sigma_0$ contains $\mathbf{O}$, and if Σ_1 is sufficiently near Σ_0, $\mathring{J}\Sigma_1$ also contains $\mathbf{O}$.

The Fréchet distance $D(\Sigma_0, \Sigma_1)$. Let u be an arbitrary point on $\mathfrak{S}$. Suppose that Σ_i, $i = 0, 1$, is given by a homeomorphic mapping $u \to \mathbf{A}_i(u)$ of $\mathfrak{S}$ onto Σ_i. Set

$$d(\mathbf{A}_0, \mathbf{A}_1) = \max_{u \in \mathfrak{S}} \| \mathbf{A}_0(u) - \mathbf{A}_1(u) \| \tag{2.1}$$

$$D(\Sigma_0, \Sigma_1) = \inf_{(\mathbf{A}_0, \mathbf{A}_1)} d(\mathbf{A}_0, \mathbf{A}_1) \tag{2.2}$$

where $\mathbf{A}_0$ and $\mathbf{A}_1$ range over all homeomorphic mappings $\mathbf{A}_0$ and $\mathbf{A}_1$ of $\mathfrak{S}$ onto Σ_0 and Σ_1 respectively.

A projection π into $\mathfrak{S}$. Corresponding to any point $\mathbf{x} \in E$ such that $\mathbf{x} \neq \mathbf{O}$, let $\pi(\mathbf{x})$ be the point of $\mathfrak{S}$ in which $\mathfrak{S}$ meets the ray from $\mathbf{O}$ through $\mathbf{x}$. Corresponding to a homeomorphic mapping $u \to \mathbf{A}_i(u)$ of $\mathfrak{S}$ onto Σ_i, a continuous mapping $u \to f_i(u)$ of $\mathfrak{S}$ into $\mathfrak{S}$ is defined by setting

$$f_i(u) = \pi(\mathbf{A}_i(u)) \qquad (u \in \mathfrak{S},\ i = 0, 1). \tag{2.3}$$

Essential maps of $\mathfrak{S}$ into $\mathfrak{S}$. A map (continuous) of $\mathfrak{S}$ into $\mathfrak{S}$ is termed essential if it is not homotopic, among maps from $\mathfrak{S}$ into $\mathfrak{S}$, to a constant map. The following results concerning essential maps are well known. A necessary and sufficient condition that $\overset{\circ}{J}\Sigma_i$, $i = 1, 2$, contain $\mathbf{O}$ is that f_i be essential. Ref. 11, p. 97. If f_0 is homotopic to f_1, then f_0 and f_1 are both essential, or both inessential.

Let $\|\Sigma_0\|$ be the minimum distance of Σ_0 from the origin $\mathbf{O}$.

Lemma 2.1. *If $\overset{\circ}{J}\Sigma_0$ contains $\mathbf{O}$, and if*

$$D(\Sigma_0, \Sigma_1) < \|\Sigma_0\|, \tag{2.4}$$

then f_1 is essential, and $\overset{\circ}{J}\Sigma_1$ contains $\mathbf{O}$.

If (2.4) holds, then for suitable choice of representations $u \to \mathbf{A}_i(u)$ of Σ_i, $i = 0, 1$,

$$d(\mathbf{A}_0, \mathbf{A}_1) < \|\Sigma_0\|. \tag{2.5}$$

Consistent with the definition of $\mathbf{A}_i$, $i = 0, 1$, set

$$\mathbf{A}_t(u) = t\mathbf{A}_1(u) + (1-t)\mathbf{A}_0(u) \qquad (0 \leq t \leq 1,\ u \in \mathfrak{S}). \tag{2.6}$$

Since (2.5) holds, no point $\mathbf{A}_t(u)$ is $\mathbf{O}$. One can accordingly define a continuous family of maps $u \to f_t(u)$ of $\mathfrak{S}$ into $\mathfrak{S}$ by setting

$$f_t(u) = \pi(\mathbf{A}_t(u)) \qquad (0 \leq t \leq 1,\ u \in \mathfrak{S}). \tag{2.7}$$

The mapping f_1 is thus homotopic to f_0. Since f_0 is essential, f_1 is essential, and $\overset{\circ}{J}\Sigma_1$ contains $\mathbf{O}$.

The proof of Lemma 2.2 is similar to that of Lemma 2.1. If Y is a subset of E the complement of Y, relative to E, is denoted by CY.

Lemma 2.2. *If $CJ\Sigma_0$ contains $\mathbf{O}$, and if 2.4 holds, then f_1 is inessential, and $CJ\Sigma_1$ contains $\mathbf{O}$.*

Proof of Lemma 1.1. That the conditions of Lemma 1.1 are necessary follows from remarks preceding the lemma. We note also that the conditions of the lemma imply that for each $p \in \Gamma$, F^p maps X^p homeomorphically onto an open subset of E. For it follows from Corollary 2 of Ref. 8, § 10 No. 4, that F^p is locally a homeomorphism, and so, by the Brouwer theorem on invariance of domain under homeomorphic mappings into E of open subsets of E, F^p maps X^p homeomorphically onto an *open* subset of E.

Assuming the conditions of Lemma 1.1 it remains to establish the continuity of F^{-1}.

Let $(\mathbf{a}, q)$ be a point of X and $(\mathbf{b}, q)$ its image in $E \times \Gamma$ under F. Since X is open there exists a neighborhood of $(\mathbf{a}, q)$ relative to X of the form $\bar{B} \times V$, where B is an open spherical n-ball in E with center $\mathbf{a}$, and V a neighborhood of q relative to Γ. Let $\mathfrak{S}$ be the $(n-1)$-sphere bounding B, and for each $p \in V$ set

$$F^p(\mathfrak{S}) = M_p . \tag{2.8}$$

Then

$$\mathring{J} M_p = F^p(B) \tag{2.9}$$

since the right member of (2.9) is a bounded open subset of E whose boundary is M_p.

Without loss of generality we can suppose that $\mathbf{a} = \mathbf{b} = \mathbf{O}$. Let B' be an n-ball with center at $\mathbf{O}$ and closure in $\mathring{J} M_q$. If $V' \subset V$ is a sufficiently small open neighborhood of q relative to Γ, then, for $p \in V'$, $D(M_p, M_q)$ will be so small that B' will not meet M_p, and, in accord with Lemma 2.1, the center $\mathbf{O}$ of the ball B' will be contained in $\mathring{J} M_p$. Since B' is connected, and does not meet M_p, we conclude that $B' \subset \mathring{J} M_p$ for each $p \in V'$. This inclusion and (2.9) imply that

$$(F^{-1})^p(B') \subset (F^{-1})^p(\mathring{J} M_p) = B .$$

Since $V' \subset V$ and F is p-invariant, it follows that

$$F^{-1}(B' \times V') \subset F^{-1}(\mathring{J} M_p \times V) = B \times V . \tag{2.10}$$

Now $B \times V$ is an arbitrarily small neighborhood of $(\mathbf{a}, q)$ relative to X. Since $B' \times V'$ is a neighborhood of $(\mathbf{b}, q)$ relative to X the inclusion (2.10) implies the continuity of F^{-1} at $(\mathbf{b}, q)$.

This completes the proof of Lemma 1.1.

As shown in the last paragraph of the proof of Lemma 1.1, the set $F(X)$ includes the open neighborhood $B' \times V'$ of the arbitrary point $(\mathbf{b}, q)$ of $F(X)$ and is accordingly open. Hence we can affirm the following generalization of the Brouwer theorem on the invariance of domain under homeomorphic mappings of open subsets of E into E.

Lemma 2.3. *If X is an open subset of $E \times \Gamma$ and F a*

p-invariant homeomorphism of X into $E \times \Gamma$, then the image $F(X)$ is open relative to $E \times \Gamma$.

Proof of Lemma 1.2. That the conditions of Lemma 1.2 are necessary is immediate.

Assume then that the conditions of Lemma 1.2 are satisfied. That the mapping F is then a homeomorphism is a consequence of Lemma 1.1. It remains to show that F is locally a C^m-diffeomorphism.

Let $(\mathbf{a}, q)$ be a point of X. Let a neighborhood of q relative to Γ_m be referred to admissible local coordinates $(v_1, \ldots, v_r)$ ranging over an open neighborhood V of the origin in an euclidean r-space. Set $(v_1, \ldots, v_r) = \mathbf{v}$. Let $p(\mathbf{v})$ be the point $p \in \Gamma_m$ represented by $\mathbf{v} \in V$. If V is sufficiently small, and if U is a sufficiently small open neighborhood of $\mathbf{a}$ relative to E, then $U \times p(V) \subset X$, and F_1 admits, by hypothesis, a representation over $U \times V$ of the form,

$$F_1(\mathbf{x}, p(\mathbf{v})) = \mathbf{h}(\mathbf{x}, \mathbf{v}) \quad ((\mathbf{x}, \mathbf{v}) \in U \times V), \tag{2.11}$$

where $\mathbf{h}$ is a vector-valued function of class C^m with

$$\frac{D(h_1, \ldots, h_n)}{D(x_1, \ldots, x_n)} \neq 0, \qquad ((\mathbf{x}, \mathbf{v}) \in U \times V). \tag{2.12}$$

The restriction of the p-invariant mapping $(\mathbf{x}, p) \to F(\mathbf{x}, p)$ to the subset $U \times p(V)$ of X can be represented in the form

$$\begin{cases} x'_i = h_i(\mathbf{x}, \mathbf{v}) & (i = 1, \ldots, n) \\ v'_j = v_j & (j = 1, \ldots, r) \end{cases} \tag{2.13}$$

in terms of the points $(\mathbf{x}, \mathbf{v}) \in U \times V$ and the coordinates of their images $(\mathbf{x}', \mathbf{v}')$. Because of (2.12) the transformation (2.13) has a non-vanishing jacobian on the coordinate range $U \times V$.

Lemma 1.2 follows.

Lemma 2.1 and the argument used in proving Lemma 1.1 imply a lemma frequently used in Part II.

Lemma 2.4. Let $\mathfrak{M}$ be a topological $(n-1)$-sphere in E and Γ a topological space. Let F be a p-invarint homeomorphism of $\mathfrak{M} \times \Gamma$ into $E \times \Gamma$. Then a subset Y of $E \times \Gamma$ such that

$$Y^p = \mathring{J} F^p(\mathfrak{M}) \qquad (p \in \Gamma) \tag{2.14}$$

is open relative to $E \times \Gamma$, and a subset W of $E \times \Gamma$ such that

$$W^p = CJF^p(\mathfrak{M}) \qquad (p \in \Gamma) \tag{2.15}$$

is also open relative to $E \times \Gamma$.

Let $(\mathbf{a}, q)$ be a point of Y. Lemma 2.1 and the argument used in proving Lemma 1.1 show that if B is an n-ball in E with center at $\mathbf{a}$ and closure in $\overset{\circ}{J}F^q(\mathfrak{M})$, and if, corresponding to B, V is chosen as a sufficiently small neighborhood of q relative to Γ, then the set $B \times V$ is included in Y. Thus Y is open.

That W is open is proved similarly, using Lemma 2.2 instead of Lemma 2.1.

We conclude § 2 by recording certain notational relations used throughout this paper.

Let g be a p-invariant mapping of a subset A of $E \times \Gamma_m$ into $E \times \Gamma_m$. Then

$$g(\mathbf{x}, p) = (g^p(\mathbf{x}), p) \qquad (\mathbf{x} \in A^p, p \in \mathrm{pr}_2 A), \tag{2.16}$$

and

$$[g(A)]^p = g^p(A^p) \qquad (p \in \mathrm{pr}_2 A). \tag{2.17}$$

Relation (2.16), it will be recalled, is the relation defining g^p over A^p for fixed $p \in \mathrm{pr}_2 A$. Relation (2.17) is verified by noting that for $p \in \mathrm{pr}_2 A$, $[g(A)]^p = g_1(A^p, p)$ by the p-invariance of g, and $g^p(A^p) = g_1(A^p, p)$ by (1.5).

If, in addition, g is biunique, one has the relation

$$(g^p)^{-1}(\mathbf{y}) = (g^{-1})^p(\mathbf{y}) \qquad (\mathbf{y} \in g^p(A^p), p \in \mathrm{pr}_2 A). \tag{2.18}$$

When compositions of mappings are used, further notational laws enter. Lemma 2.5 below, which states a fundamental principle for such compositions, makes use of a convention which we shall adopt in this paper.

Convention. Let F' and F be C^m-diffeomorphisms, $m \geq 0$, into E of open subsets, X' and X, respectively, of E, such that $F(X)$ meets X'. We understand that $F'F$ is defined on the subset $F^{-1}(X')$ of X.

A similar convention will be adopted when F' and F are p-invariant C^m-diffeomorphisms, $m \geq 0$, into $E \times \Gamma_m$ of open subsets of $E \times \Gamma_m$.

The domain of definition $F^{-1}(X')$ prescribed for the composite mapping $F'F$ will be termed canonical. It is an open subset of E (respectively, $E \times \Gamma_m$). The mapping $F'F$ is a C^m-diffeomorphism of $F^{-1}(X')$.

Two cases arise according as X' includes $F(X)$ or does not include $F(X)$. The first case is the classical case. Here, the canonical domain $F^{-1}(X')$ of $F'F$ is X. This case arose, for example, in defining $f\varphi$ on N in (1.11). In the second case $F^{-1}(X')$ is a proper open subset of X. The second case arises in this paper in §§ 3, 4, 5, 16.

The following lemma will be referred to as the Notational Lemma.

LEMMA 2.5. *Let F and f be two p-invariant homeomorphisms into $E \times \Gamma_m$ of subsets X and Y, respectively, of $E \times \Gamma_m$. Let V be the canonical domain of definition of Ff in $E \times \Gamma_m$, and let W be the subset of $E \times \Gamma_m$ whose p-section W^p is the canonical domain of definition of $F^p f^p$. Then $V = W$, and*

$$(Ff)^p(\mathbf{x}) = (F^p f^p)(\mathbf{x}) \qquad ((\mathbf{x}, p) \in W). \tag{2.19}$$

Equivalently, (2.19) can be written in the form

$$(Ff)_1(\mathbf{x}, p) = (F^p f^p)(\mathbf{x}) \qquad ((\mathbf{x}, p) \in W). \tag{2.20}$$

By definition of V and W^p the relation $V = W$ becomes the relation

$$[f^{-1}(X)]^p = (f^p)^{-1}(X^p). \tag{2.21}$$

This relation is a consequence of (2.18) when g is replaced by f.

To establish (2.19) we apply (2.16) three times, replacing g successively by Ff, f and F. We find that for $(\mathbf{x}, p) \in W$

$$((Ff)^p(\mathbf{x}), p) = (Ff)(\mathbf{x}, p) = F(f^p(\mathbf{x}), p) = (F^p(f^p(\mathbf{x})), p)$$

from which (2.19) follows.

The equivalence of (2.19) and (2.20) is a consequence of the definition of $(fF)^p$.

The canonical domain $V = W$ is *maximal* in the sense of the following corollary of the lemma.

COROLLARY 2.1. *If $(\mathbf{x}, p)$ is a point in $E \times \Gamma_m$ such that $F^p(f^p(\mathbf{x}))$ or $F(f(\mathbf{x}, p))$ is defined, then $(\mathbf{x}, p)$ is in $V = W$. In particular $(\mathbf{x}, p)$ is in the domain of $(Ff)_1$.*

§ 3. External transformations of simple problems.

Let $P = (\varphi, N, m)$ be a simple problem. If (Λ_φ, N_*) is a solution of P, Λ_φ is defined on $N \cup \mathring{J} S$, and the set

$$\Lambda_\varphi (N \cup \mathring{J} S) = \varphi(N) \cup \mathring{J} \varphi(S) = W(P) \tag{3.1}$$

is an open subset of E which is completely determined by P. If f is a C^m-diffeomorphism into E of an open subset U of E which includes $W(P)$, we shall say that f *operates externally* on P and define fP by setting

$$fP = (f\varphi, N, m). \tag{3.2}$$

We understand thereby that $f\varphi$ is defined on N, in accord with our convention in § 2.

In Ref. 4 the following is shown.

I. $(f\varphi, N, m)$ *defines a simple Schoenflies problem.*

The proof of I, as given in Ref. 4, recognizes that $f\varphi$ defines a C^m-diffeomorphism of N, and shows that $f\varphi$ maps points on N interior (exterior) to S into points of E interior (exterior) to $(f\varphi)(S)$ and that

$$f(\mathring{J}\varphi(S)) = \mathring{J}(f\varphi)(S). \tag{3.3}$$

The proof of Lemma 1 (ii), Ref. 4, then implies the following.

Theorem 3.1. *If (Λ_φ, N_*) is a solution of the simple problem P, then with $f\Lambda_\varphi$ understood as defined on $W(P)$, $(f\Lambda_\varphi, N_*)$ is a solution of the simple problem fP.*

We term the simple problem fP the image of P under the *external operator* f.

We shall prove the following.

II. *If f operates externally on a simple problem P then*

$$W(fP) = f(W(P)). \tag{3.4}$$

In accord with the definition of $W(fP)$

$$W(fP) = (f\varphi)(N) \cup \mathring{J}(f\varphi)(S) = f(\varphi(N) \cup \mathring{J}\varphi(S)) = f(W(P)) \tag{3.5}$$

thus verifying (3.4).

Use will be made of (3.4) to prove the following.

III. If f operates externally on P, then f^{-1} operates externally on fP.

We must show that f^{-1} defines a C^m-diffeomorphism into E of $W(fP)$. Since f^{-1} defines a C^m-diffeomorphism of $f(W(P))$ and $f(W(P)) = W(fP)$, f^{-1} operates externally on fP as stated.

With III established, Theorem 3.1 has the corollary:

Corollary 3.1. If $(\Lambda_{f\varphi}, N_*)$ is a solution of the problem fP then

$$(f^{-1}\Lambda_{f\varphi}, N_*) \tag{3.6}$$

is a solution of the problem $f^{-1}(fP) = P$.

If f operates externally on the simple problem P, we say that P and fP are externally equivalent under f. When an affirmation of external equivalence of P to fP is made, it is always understood that f defines a C^m-diffeomorphism of $W(P)$ into E (by restriction of f to $W(P)$, if necessary). The integer m is the integer m in the triple (φ, N, m) defining P.

IV. If f operates externally on P and f' on fP, then $f'f$ operates externally on P and

$$(f'f)P = f'(fP). \tag{3.7}$$

The domain of definition of f' includes $W(fP)$ by hypothesis. The domain of definition of $f'f$, by our convention, then includes

$$f^{-1}(W(f(P))) = f^{-1}(f(W(P))) = W(P) \tag{3.8}$$

so that $f'f$ operates externally on P. We remark that $W(fP)$ need not include the range of f, in fact the range of f might be all of E.

One verifies (3.7) formally, using the definition (3.2).

It is seen that the relation of external equivalence in the class of simple problems is reflexive, symmetric and transitive. Simple problems P^* and P are equivalent if and only if there exists an f which operates externally on P, and is such that $P^* = fP$, or equivalently, if f^{-1} operates externally on P^* and $f^{-1}P^* = P$.

If f operates on P, the neighborhood N and class index m are invariant in the transformation $P \to fP$. The nature of an internal transformation of P will be found to be somewhat different.

§ 4. External transformation of Γ_m-problems.

Given the Γ_m-problem $\mathbf{P} = (\Phi , L , \Gamma_m)$, we first define an analogue of $W(P)$ of § 3.

The set $\mathbf{W(P)}$. If (Λ_Φ , L_*) is a solution of $\mathbf{P}$, then the set

$$Y = L \cup (\mathring{J}S \times \Gamma_m)$$

is the domain of definition of Λ_Φ, and one can set

$$\Lambda_\Phi(Y) = \mathbf{W(P)}. \tag{4.1}$$

This definition of $\mathbf{W(P)}$ is not adequate because we shall need $\mathbf{W(P)}$ before we have a priori knowledge that a solution of $\mathbf{P}$ exists, and because it is not clear from the definition (4.1) that $\mathbf{W(P)}$ depends only on $\mathbf{P}$. The definition (4.2) of $\mathbf{W(P)}$, which we adopt, agrees with (4.1) when a solution of $\mathbf{P}$ exists. We set

$$\mathbf{W(P)} = \bigcup_{p \in \Gamma_m} (\mathring{J}\Phi^p(S) \times p) \cup \Phi(L). \tag{4.2}$$

So defined $\mathbf{W(P)}$ depends only on $\mathbf{P}$.

The representation (4.1) of $\mathbf{W(P)}$ had the merit of showing, in accord with the generalized Brouwer theorem, Lemma 2.3, that $\mathbf{W(P)}$ is an open subset of $E \times \Gamma_m$, granting that a solution of $\mathbf{P}$ exists. However the same result follows directly from (4.2) without assuming any solution of $\mathbf{P}$. For $\Phi(L)$ is open relative to $E \times \Gamma_m$ by Lemma 2.3, while the set

$$\bigcup_{p \in \Gamma_m} (\mathring{J}\Phi^p(S) \times p) \tag{4.3}$$

is open relative to $E \times \Gamma_m$ by Lemma 2.4.

An external operator F. Let F be a p-invariant C^m-diffeomorphism into $E \times \Gamma_m$ of an open subset of $E \times \Gamma_m$ which includes $\mathbf{W(P)}$. We shall then say that F *operates externally* on $\mathbf{P}$ and set

$$F\mathbf{P} = (F\Phi , L , \Gamma_m). \tag{4.4}$$

The following statements will now be proved.

I. $(F\Phi , L , \Gamma_m)$ *defines a* Γ_m-*problem.*

We refer to the simple problem

$$\mathbf{P}^p = (\Phi^p , L^p , m) \tag{4.5}$$

defined by $\mathbf{P}$ for each $p \in \Gamma_m$. From definitions (4.2) and (3.1) it follows that

$$[\mathbf{W}(\mathbf{P})]^p = \overset{\circ}{J}\Phi^p(S) \cup \Phi^p(L^p) = W(\mathbf{P}^p) \qquad (p \in \Gamma_m). \tag{4.6}$$

To establish I one first recognizes that $F\Phi$ defines a C^m-diffeomorphism of L into $E \times \Gamma_m$, since F is defined over $\mathbf{W}(\mathbf{P})$ and $\mathbf{W}(\mathbf{P}) \supset \Phi(L)$. Statement I will follow if we show that for each $p \in \Gamma_m$, $(F\Phi)^p$ maps points of L^p which are interior (exterior) to S into points which are interior (exterior) to the topological $(n-1)$-sphere $(F\Phi)^p(S)$, or equivalently if we show that

$$(F\mathbf{P})^p = ((F\Phi)^p, L^p, m) = (F^p \Phi^p, L^p, m) \tag{4.7}$$

defines a simple problem. This will follow from I of § 3, if we show that F^p operates externally on $\mathbf{P}^p$, that is, if F^p is defined over $W(\mathbf{P}^p)$. But F^p is defined over $W(\mathbf{P}^p)$ since F is defined over $\mathbf{W}(\mathbf{P})$ and (4.6) holds. Statement I follows.

II. *If F operates externally on $\mathbf{P}$ then*

$$\mathbf{W}(F\mathbf{P}) = F\mathbf{W}(\mathbf{P}). \tag{4.8}$$

For each $p \in \Gamma_m$

$$\begin{aligned}[\mathbf{W}(F\mathbf{P})]^p &= W((F\mathbf{P})^p) = W(F^p \mathbf{P}^p) = F^p(W(\mathbf{P}^p)) \\ &= F^p[(\mathbf{W}(\mathbf{P}))^p] = [F(\mathbf{W}(\mathbf{P}))]^p.\end{aligned} \tag{4.9}$$

One uses (4.6) to justify the first and fourth equalities, and (3.4) to justify the third. We have seen that F^p operates externally on $\mathbf{P}^p$, in the proof of I. The second and fifth equalities follow respectively from the definition of $(F\mathbf{P})^p$ by means of (4.7), and the p-invariance of F. Relation (4.8) follows from (4.9).

Theorem 4.1 can now be proved.

Theorem 4.1. *If (Λ_Φ, L_*) is a solution of the Γ_m-problem $\mathbf{P}$, then*

$$(F\Lambda_\Phi, L_*) \tag{4.10}$$

is a solution of the Γ_m-problem $F\mathbf{P}$.

Recall that Λ_Φ is defined on the above set Y, that $\Lambda_\Phi(Y) = \mathbf{W}(\mathbf{P})$, and that F is defined on $\mathbf{W}(\mathbf{P})$. Thus $F\Lambda_\Phi$ is a p-invariant homeomorphism of Y into $E \times \Gamma_m$. Since Λ_Φ extends $\Phi \mid (L_e \cup L_*)$, $F\Lambda_\Phi$ extends $(F\Phi) \mid (L_e \cup L_*)$. If $m > 0$, Λ_Φ is in addition a C^m-diffeomorphism of the set

$$X = L \cup (J_0 S \times \Gamma_m) \subset Y \tag{4.11}$$

into $E \times \Gamma_m$, so that $F\Lambda_\Phi$ is a C^m-diffeomorphism of X into $E \times \Gamma_m$. This establishes Theorem 4.1.

Statement II implies III.

III. If F operates externally on $\mathbf{P}$, then F^{-1} operates externally on $F\mathbf{P}$.

With III established it is evident that Theorem 4.1 has the following corollary.

Corollary 4.1. If $(\Lambda_{F\Phi}, L_*)$ is a solution of the Γ_m-problem $F\mathbf{P}$, then

$$(F^{-1}\Lambda_{F\Phi}, L_*)$$

is a solution of the problem $F^{-1}(F\mathbf{P}) = \mathbf{P}$.

If F operates externally on a Γ_m-problem $\mathbf{P}$ we say that $\mathbf{P}$ and $F\mathbf{P}$ are externally equivalent under F. It must be understood that the affirmation that F operates externally on a Γ_m-problem $\mathbf{P}$, implies that F defines a p-invariant C^m-diffeomorphism of $\mathbf{W}(\mathbf{P})$ into $E \times \Gamma_m$ (by restriction of F to $\mathbf{W}(\mathbf{P})$, if necessary).

IV. If F operates externally on $\mathbf{P}$ and f on $F\mathbf{P}$, then fF operates externally on $\mathbf{P}$ and

$$f(F\mathbf{P}) = (fF)\mathbf{P}. \tag{4.12}$$

Using the homogeneity relation (4.8), the proof that fF operates externally on $\mathbf{P}$ is similar to the proof of IV in § 3. Use is made of our convention on the domain of fF. The relation (4.12) is verified formally. Here, as always, one distinguishes between the question of the admissibility of operations indicated, and the formal verification of equalities such as (4.12).

A necessary and sufficient condition that two Γ_m-problems $\mathbf{P}$ and $\mathbf{P}^*$ be externally equivalent is that there exist a mapping F which operates externally on $\mathbf{P}$ and $\mathbf{P}^* = F\mathbf{P}$, or equivalently that F^{-1} operates externally on $\mathbf{P}^*$ and $\mathbf{P} = F^{-1}\mathbf{P}^*$. With this formulation of external equivalence it is easy to verify that this relation is reflexive, symmetric and transitive.

§5. Internal transformations of Γ_m-problems.

In the case of simple problems the notion of external equivalence is sufficient for our purposes. In the case of Γ_m-problems the notion of external equivalence needs to be supplemented by the concept of internal equivalence. This we now define.

Reduced mappings F of $E\times\Gamma_m$ onto $E\times\Gamma_m$. By a reduced C^m-diffeomorphism f of E onto E we mean a C^m-diffeomorphism of E onto E which leaves fixed the center Z of S and the point $Q=(0, 0, ..., 0, 1)$, and leaves S invariant as a set. By a reduced mapping F of $E\times\Gamma_m$ onto $E\times\Gamma_m$ we mean a p-invariant C^m-diffeomorphism of $E\times\Gamma_m$ onto $E\times\Gamma_m$ such that for each $p\in\Gamma_m$, F^p is a reduced C^m-diffeomorphism of E onto E. Corresponding to an arbitrary Γ_m-problem $\mathbf{P}=(\Phi, L, \Gamma_m)$ and a reduced mapping F of $E\times\Gamma_m$ onto $E\times\Gamma_m$ we set

$$\mathbf{P}F=(\Phi F, F^{-1}(L), \Gamma_m), \tag{5.1}$$

understanding that ΦF is defined on $F^{-1}(L)$.

Reduced problems. A simple problem (φ, N, m) will be termed reduced if $\varphi(Q)=Q$. A Γ_m-problem $\mathbf{P}=(\Phi, L, \Gamma_m)$ will be termed reduced if $\Phi^p(Q)=Q$ for each $p\in\Gamma_m$.

We shall prove the following.

I. *If F is a reduced mapping of $E\times\Gamma_m$ onto $E\times\Gamma_m$, and $\mathbf{P}=(\Phi, L, \Gamma_m)$ a Γ_m-problem, then $\mathbf{P}F$ is a Γ_m-problem. If $\mathbf{P}$ is reduced, then $\mathbf{P}F$ is reduced.*

The set $F^{-1}(L)$ is an open neighborhood of $S\times\Gamma_m$ relative to $(E-Z)\times\Gamma_m$, since L is an open neighborhood of $S\times\Gamma_m$ relative to $(E-Z)\times\Gamma_m$, since both $Z\times\Gamma_m$ and $S\times\Gamma_m$ are invariant as sets under F^{-1}, and since $F^{-1}(L)$ is open (Lemma 2.3) relative to $E\times\Gamma_m$. Moreover, for each $p\in\Gamma_m$, $(\Phi F)^p=\Phi^p F^p$ maps points of $[F^{-1}(L)]^p=(F^{-1})^p(L^p)$ intetior (exterior) to S, into points interior (exterior) to $(\Phi F)^p(S)$. This is a consequence of the fact that F^p leaves JS and its complement setwise invariant, and Φ^p maps points of L^p interior (exterior) to S into points interior (exterior) to $\Phi^p(S)$. Moreover ΦF is a p-invariant C^m-diffeomorphism of $F^{-1}(L)$ into $E\times\Gamma_m$. Thus $\mathbf{P}F$ is a Γ_m-problem.

Finally $\mathbf{P}F$ is reduced if $\mathbf{P}$ is reduced, since both Φ^p and F^p then leave Q fixed for each $p \in \Gamma_m$.

Theorem 5.1. *Let F be a reduced mapping of $E \times \Gamma_m$ onto $E \times \Gamma_m$ and $\mathbf{P} = (\Phi, L, \Gamma_m)$ a Γ_m-problem. If (Λ_Φ, L_*) is a solution of $\mathbf{P}$ then $(\Lambda_\Phi F, F^{-1}(L_*))$ is a solution of $\mathbf{P}F$.*

The mapping $\Lambda_\Phi F$ is a p-invariant homeomorphism into $E \times \Gamma_m$ of the open set

$$F^{-1}(L \cup (JS \times \Gamma_m)) = F^{-1}(L) \cup (JS \times \Gamma_m) \tag{5.2}$$

on which $\Lambda_\Phi F$ is understood to be defined. Moreover $\Lambda_\Phi F$ extends

$$(\Phi F) \mid F^{-1}(L_e \cup L_*) = (\Phi F) \mid [(F^{-1}(L))_e \cup F^{-1}(L_*)]\,. \tag{5.3}$$

When $m > 0$, $\Lambda_\Phi F$ is, in addition, a C^m-diffeomorphism into $E \times \Gamma_m$ of

$$F^{-1}[L \cup (J_0 S \times \Gamma_m)] = F^{-1}(L) \cup (J_0 S \times \Gamma_m)\,. \tag{5.4}$$

This establishes Theorem 5.1.

Corollary 5.1. *If L_* is an open neighborhood of $S \times \Gamma_m$ in $E \times \Gamma_m$ such that $(\Lambda_{\Phi F}, F^{-1}(L_*))$ is a solution of the problem $\mathbf{P}F$ of Theorem 5.1, then*

$$(\Lambda_{\Phi F} F^{-1}, L_*) \tag{5.5}$$

is a solution of problem $\mathbf{P}$.

Internal equivalence of Γ_m-problems. By virtue of Theorem 5.1 and its corollary we say that a Γ_m-problem $\mathbf{P} = (\Phi, L, \Gamma_m)$ and $\mathbf{P}F$ are *internally equivalent under* F, whenever F is a reduced mapping of $E \times \Gamma_m$ onto $E \times \Gamma_m$. It is understood always that a reduced mapping of $E \times \Gamma_m$ onto $E \times \Gamma_m$ is a reduced p-invariant C^m-diffeomorphism of $E \times \Gamma_m$ onto $E \times \Gamma_m$. We say that such an F *operates internally* on $\mathbf{P}$.

In the class of Γ_m-problems with a common Γ_m the relation of internal equivalence is reflexive, symmetric, and transitive.

§ 6. General equivalence of Γ_m-problems.

A reduced mapping F of $E \times \Gamma_m$ onto $E \times \Gamma_m$, as defined in § 5, operates internally on an *arbitrary* Γ_m-problem $\mathbf{P}$, producing a Γ_m-problem $\mathbf{P}F$. On the other hand a mapping f which is an external operator on a

Γ_m-problem $\mathbf{P}$ must define a C^m-diffeomorphism of $\mathbf{W}(\mathbf{P})$ into $E \times \Gamma_m$, and will not be, a priori, an external operator on a Γ_m-problem different from $\mathbf{P}$. However, we shall prove the following.

I. (i) If f operates externally on a Γ_m-problem $\mathbf{P}$, and if F is an arbitrary reduced mapping of $E \times \Gamma_m$ onto $E \times \Gamma_m$, then f operates externally on $\mathbf{P}F$. Moreover (ii) $(f\mathbf{P})F = f(\mathbf{P}F)$.

To prove I. (i) it is sufficient to prove that

$$\mathbf{W}(\mathbf{P}) = \mathbf{W}(\mathbf{P}F)\,. \tag{6.1}$$

For f operates externally on a Γ_m-problem $\mathbf{P}^*$ if and only if it defines a p-invariant C^m-diffeomorphism of $\mathbf{W}(\mathbf{P}^*)$ into $E \times \Gamma_m$. If then (6.1) holds, f operates externally on $\mathbf{P}F$, as well as on $\mathbf{P}$.

One begins with the representations

$$\mathbf{P} = (\Phi\,, L\,, \Gamma_m) \qquad \mathbf{P}F = (\Phi F\,, F^{-1}(L)\,, \Gamma_m)\,. \tag{6.2}$$

By virtue of the definition (4.2),

$$\mathbf{W}(\mathbf{P}F) = \bigcup_{p \in \Gamma_m} [\overset{\circ}{J}(\Phi F)^p(S) \times p] \cup (\Phi F)(F^{-1}(L))\,.$$

Since F is a reduced mapping, this set reduces to

$$\bigcup_{p \in \Gamma_m} (\overset{\circ}{J}\Phi^p(S) \times p) \cup \Phi(L) = \mathbf{W}(\mathbf{P}) \tag{6.3}$$

thus verifying (6.1), and establishing I. (i).

If one makes formal use of the definition (5.1), one finds that

$$(f\mathbf{P})F = f(\mathbf{P}F)\,. \tag{6.4}$$

Note that the left member of (6.4) is well-defined, since $f\mathbf{P}$ is supposed well-defined and F is a reduced mapping of $E \times \Gamma_m$ onto $E \times \Gamma_m$. The right member is well-defined by virtue of I. (i).

General equivalence of Γ_m-problems. In the class of all Γ_m-problems with a given parameter space Γ_m an equivalence relation will now be defined.

Recall first that the inverse of a reduced mapping F of $E \times \Gamma_m$ onto $E \times \Gamma_m$ is a reduced mapping of $E \times \Gamma_m$ onto $E \times \Gamma_m$.

Definition. Two Γ_m-problems $\mathbf{P}$ and $\mathbf{P}^\circ$ will be termed equivalent if either one of two equivalent conditions (i) or (ii) is satisfied.

(i) There exists a reduced mapping F of $E\times\Gamma_m$ onto $E\times\Gamma_m$, and a mapping f which operates externally on $\mathbf{P}$, such that $\mathbf{P}^* = f\mathbf{P}F$.

(ii) There exists a reduced mapping F^* of $E\times\Gamma_m$ onto $E\times\Gamma_m$, and a mapping f^* which operates externally on $\mathbf{P}^*$ such that $\mathbf{P} = f^*\mathbf{P}^*F^*$.

The equivalence of conditions (i) and (ii) must be verified. We show that (i) implies (ii). Under (i), f^{-1} operates externally on $f\mathbf{P}$ by III of § 4. Then f^{-1} operates externally on $(f\mathbf{P})F = \mathbf{P}^*$ by I of this section. Thus $f^{-1}\mathbf{P}^*$ and $f^{-1}\mathbf{P}^*F^{-1}$ are well-defined Γ_m-problems. The relation $f^{-1}\mathbf{P}^*F^{-1} = \mathbf{P}$ is a formal consequence of the relation $\mathbf{P}^* = f\mathbf{P}F$. Thus (i) implies (ii). Similarly (ii) implies (i).

If $\mathbf{P}^*$ and $\mathbf{P}$ are equivalent in the above sense, f and F are not uniquely determined, nor are f^* and F^*, as examples will show.

We shall review the structure of operations on Γ_m-problems.

If f operates externally on a Γ_m-problem $\mathbf{P}$ and f' on $f\mathbf{P}$, then $f'f$ operates externally on $\mathbf{P}$ and $f'(f\mathbf{P}) = (f'f)\mathbf{P}$. If f operates externally on $\mathbf{P}$, f^{-1} operates externally on $f\mathbf{P}$. If f operates externally on $\mathbf{P}$ and F is an arbitrary reduced mapping of $E\times\Gamma_m$ onto $E\times\Gamma_m$, then f operates externally on $\mathbf{P}F$ and $(f\mathbf{P})F = f(\mathbf{P}F)$. If F and F' are arbitrary reduced mappings of $E\times\Gamma_m$ onto $E\times\Gamma_m$, then FF' is a reduced mapping of $E\times\Gamma_m$ onto $E\times\Gamma_m$ and $(\mathbf{P}F)F' = \mathbf{P}(FF')$. Finally (i) and (ii) are equivalent definitions of the equivalence of $\mathbf{P}$ and $\mathbf{P}^*$.

With the aid of these properties one shows that the relation of equivalence between two Γ_m-problems is reflexive, symmetric and transitive. Moreover if $\mathbf{P}$ and $\mathbf{P}^*$ are externally equivalent or internally equivalent, they are equivalent in the general sense.

We close this section with a preliminary lemma.

Lemma 6.1. *Each simple problem $P = (\varphi, N, m)$ is externally equivalent to a reduced simple problem, and each Γ_m-problem $\mathbf{P} = (\Phi . L, \Gamma_m)$ is externally equivalent to a reduced Γ_m-problem.*

Given P, let f be a translation of E onto E which carries $\varphi(Q)$ into Q. The problem fP is externally equivalent to P and is reduced.

Let $\mathbf{q}$ be the vector representing Q. Given $\mathbf{P}$ there exists a p-invariant C^m-diffeomorphism F of $E \times \Gamma_m$ onto $E \times \Gamma_m$ such that

$$F_1(\mathbf{x}, p) = \mathbf{x} - \Phi^p(\mathbf{q}) + \mathbf{q}.$$

Cf. Lemmas 1.1 and 1.2. The Γ_m-problem $F\mathbf{P}$ is externally equivalent to the problem $\mathbf{P}$ and is reduced.

§ 7. The basic reduction theorem.

Ref. 4. Let r be a positive constant, and let $B(r)$ be an open spherical n-ball in E with center at Q and radius r. We shall make use of such an n-ball throughout this paper. We refer to the simple reduced problem $P = (\varphi, N, m)$.

Accessory constants c_1, c_2, c_3. We shall correlate the proof of Theorem 7.1 with two positive constants c_1 and c_2, termed *accessory* to the problem P. A third positive constant c_3 will be defined and used later. These constants and their extensions for reduced Γ_m-problems are central in the proof that an arbitrary Γ_m-problem is equivalent to a *uniform* Γ_m-problem (defined in § 8).

Definition of c_1. Let c_1 be a positive constant such that

$$B(2c_1) \subset \varphi(N). \tag{7.1}$$

We term c_1 *directly accessory* to P.

Definition of c_2. Let c_1 be a constant directly accessory to P. Corresponding to c_1 let c_2 be a constant such that $0 < 2c_2 \leq 1$, $B(c_2) \subset N$, and

$$\varphi(B(c_2)) \subset B(c_1). \tag{7.2}$$

We term c_2 *inversely accessory to* P, and say that c_2 is *dominated* by c_1.

Lemma 2 of Ref. 4 will be given the following form.

Theorem 7.1. *Let $P = (\varphi, N, m)$ be an arbitrary simple reduced problem to which a constant c_2 is inversely accessory. The problem P is then externally equivalent to a problem $fP = (f\varphi, N, m)$ such that*

$$(7.3) \qquad (f\varphi)(\mathbf{x}) = \mathbf{x} \qquad (\mathbf{x} \in B(c_2)).$$

Suppose c_2 dominated by c_1, a constant directly accesory to P. We introduce certain mappings.

The mapping λ. Let $t \to \lambda(t)$ be a C^∞-diffeomorphism of $(0, +\infty)$ onto $(0, 2c_1)$, such that $\lambda(t) = t$ for $t \in (0, c_1)$. Such a mapping exists.

The mapping μ. Let μ be the C^∞-diffeomorphism of E onto $B(2c_1)$, such that the point $\mathbf{q} = (0, \ldots, 0, 1) \in E$ is fixed, and such that for $\mathbf{y} \in E - \mathbf{q}$

$$(7.4) \qquad \mathbf{y} \to \mu(\mathbf{y}) = \frac{\mathbf{y} - \mathbf{q}}{\|\mathbf{y} - \mathbf{q}\|} \lambda(\|\mathbf{y} - \mathbf{q}\|) + \mathbf{q}.$$

The mapping μ leaves $\mathbf{q}$ fixed and reduces to the identity for $\mathbf{y} \in B(c_1)$.

The external operator f. Set

$$(7.5) \qquad f(\mathbf{y}) = \varphi^{-1}(\mu(\mathbf{y})) \qquad (\mathbf{y} \in E).$$

The mapping f is well-defined, since $\mu(E) = B(2c_1)$ and since φ^{-1} is defined on $B(2c_1)$. Cf. (7.1). Moreover f is a C^m-diffeomorphism over E. Note that $f(\mathbf{y}) = \varphi^{-1}(\mathbf{y})$ for $\mathbf{y} \in B(c_1)$, in accord with the definition of μ. One thus has

$$(7.6) \qquad (f\varphi)(\mathbf{x}) = \mathbf{x} \qquad (\mathbf{x} \in \varphi^{-1}(B(c_1))).$$

Now (7.2) implies that

$$B(c_2) \subset \varphi^{-1}(B(c_1))$$

so that (7.6) holds for $\mathbf{x} \in B(c_2)$. Theorem 7.1 follows.

Definition. A reduced problem $P = (\varphi, N, m)$ with which there is associated an $(n-1)$-sphere S_Q with center Q such that

$$(7.7) \qquad \varphi(\mathbf{x}) = \mathbf{x} \qquad (\mathbf{x} \in \mathring{J} S_Q)$$

will be termed e l e m e n t a r y.

Lemma 6.1 and Theorem 7.1 have the following corollary.

Corollary 7.1. *Corresponding to an arbitrary simple problem there exists an externally equivalent elementary Schoenflies problem.*

Corresponding to any positive constant ρ we introduce the spherical shell

(7.8)
$$s_\rho = \left\{\mathbf{x} \mid \frac{1}{1+\rho} < \|\mathbf{x}\| < 1+\rho\right\}.$$

The accessory constant c_3. A constant c_3 such that

(7.9)
$$0 < c_3 < 1\,, \qquad s_{c_3} \subset N$$

will be termed radially accessory to the problem $P = (\varphi\,, N\,, m)$.

Maxima of accessory constants. If $c^{(1)}, \ldots, c^{(r)}$ is a finite sequence of constants directly accessory to the problem P then the constant

(7.10)
$$c_1 = \max_i c^{(i)}$$

is directly accessory to the problem P. The maximum of a finite sequence of constants radially accessory to P is similarly radially accessory to the problem P. The maximum of a finite sequence of constants $a^{(i)}$, inversely accessory to P, and dominated respectively by constants $c^{(i)}$ directly accessory to P, is a constant a_2, inversely accessory to P, and dominated by the maximum of the constants $c^{(i)}$.

§ 8. Mappings $\vartheta_1, \vartheta_2, \vartheta_3$ accessory to a reduced Γ_m-problem.

When $m = 0$ it is assumed that Γ_m is an arbitrary paracompact space, and when $m > 0$, a connected differentiable manifold of class C^m with a countable base. When $m > 0$ Γ_m is locally compact with a countable base, hence "countable at infinity", Ref. 8, § 10, no. 11, and finally paracompact. Ref. 8, § 10, no. 12. We are using the term paracompact in the sense of Bourbaki.

Definition. A topological space X is said to be paracompact if it is a Hausdorff space and if every open covering of X admits an open locally finite refinement.

To avoid ambiguity we recall the following definition, Ref. 9, § 4, no. 3.

Definition. A partition of unity on a topological space X is a family $(f_i)_{i\in A}$ of numerically valued functions $f_i \geqq 0$, continuous on X, with supports forming a locally finite ensemble of sets on X, and such that

(8.1)
$$\sum_{i\in A} f_i(p) = 1 \qquad (p \in X)\,.$$

Definition. By a C^m-partition of unity on Γ_m we mean a

partition of unity on Γ_m such that each mapping f_i is of class C^m on Γ_m.

The following result is known.

Lemma 8.1. Given an open covering $\mathbf{V}$ of Γ_m there exists a C^m-partition of unity subordinate to $\mathbf{V}$.

In the case $m > 0$ this follows from results presented in Ref. 10, page 6, Cor. 2. When $m = 0$ the lemma is established in Ref. 9, § 4, no. 4.

We return to the reduced Γ_m-problem $\mathbf{P}$ of Lemma 6.1. For each $p \in \Gamma_m$ let $\vartheta_\nu(p)$, $\nu = 1, 2, 3$, be a constant directly, inversely, and radially accessory to the simple problem

$$\mathbf{P}^p = (\Phi^p, L^p, m), \tag{8.2}$$

with $\vartheta_2(p)$ dominated by $\vartheta_1(p)$.

We then term the mapping $p \succ \vartheta_\nu(p)$ of Γ_m into R^+, directly accessory to $\mathbf{P}$ if $\nu = 1$, inversely accessory if $\nu = 2$, and radially accessory if $\nu = 3$, and say that ϑ_2 is dominated by ϑ_1.

We shall prove the following lemma.

Lemma 8.2. There exists a C^m-mapping $p \succ \vartheta_1(p)$ directly accessory to a reduced Γ_m-problem $\mathbf{P}$.

Let t be an arbitrary fixed point of Γ_m. Recall that $\mathbf{q}$ is the vector representing Q. The point $(\mathbf{q}, t)$ is in the open subset $\Phi(L)$ of $E \times \Gamma_m$. There accordingly exists an open n-ball $B(2a_t)$, of positive radius $2a_t$ and center $\mathbf{q}$, and an open neighborhood V_t of t relative to Γ_m, such that

$$B(2a_t) \times V_t \subset \Phi(L). \tag{8.3}$$

Consequently

$$B(2a_t) \subset [\Phi(L)]^p = \Phi^p(L^p) \qquad (p \in V_t). \tag{8.4}$$

The inclusion (8.4) implies that, for fixed t, a_t is a constant directly accessory to the problem $\mathbf{P}^p$ for each $p \in V_t$.

Now $(V_t)_{t \in \Gamma_m}$ is an open covering $\mathbf{V}$ of Γ_m. Lemma 8.1 implies the existence of a C^m-partition of unity $(f_i)_{i \in A}$, subordinate to $\mathbf{V}$. There thus exists a mapping $i \succ t(i)$ of A into Γ_m such that the support of f_i is included in $V_{t(i)}$. Set

$$\vartheta_1(p) = \sum_{i \in A} a_{t(i)} f_i(p) \qquad (p \in \Gamma_m). \tag{8.5}$$

We shall show that the mapping $p \to \vartheta_1(p)$ so defined satisfies Lemma 8.2.

This mapping is clearly of class C^m. Moreover the relation $\sum_i f_i(p) = 1$ implies that

$$0 < \vartheta_1(p) \leq \max_i a_{t(i)} \qquad (i \in w(p)),$$

where $w(p)$ is the finite set of values of i such that the support of f_i meets p. Since each constant $a_{t(i)}$, for $i \in w(p)$, is a constant directly accessory to $\mathbf{P}^p$, $\vartheta_1(p)$ is also directly accessory to $\mathbf{P}^p$.

The mapping $p \to \vartheta_1(p)$ thus satisfies the lemma.

Lemma 8.2 leads to the following.

Lemma 8.3. *There exist C^m-mappings $p \to \vartheta_\nu(p)$, $\nu = 1, 2, 3$, directly, inversely and radially accessory, respectively, to a reduced Γ_m-problem $\mathbf{P}$, of which $p \to \vartheta_1(p)$ is an arbitrary C^m-mapping directly accessory to $\mathbf{P}$, and ϑ_2 is dominated by ϑ_1.*

The proof of Lemma 8.3 is similar to the proof of Lemma 8.2. Given a directly accessory C^m-mapping $p \to \vartheta_1(p)$, one defines $\vartheta_2(p)$ with the aid of a partition of unity so that $\vartheta_2(p)$ is dominated by $\vartheta_1(p)$ and $0 < 2\vartheta_2(p) \leq 1$, while the mapping $p \to \vartheta_2(p)$ is of class C^m and inversely accessory to $\mathbf{P}$. The proof of the existence of the mapping $p \to \vartheta_3(p)$ is similar to the proof in Lemma 8.2 of the existence of the mapping $p \to \vartheta_1(p)$.

Uniform Γ_m-problems. *A reduced Γ_m-problem $\mathbf{P}$ for which there exist constant accessory mappings $p \to \vartheta_\nu(p)$, $\nu = 1, 2, 3$ with ϑ_2 dominated by ϑ_1, will be called a uniform Γ_m-problem.*

Theorem 7.1 will be generalized for any uniform Γ_m-problem $\mathbf{P}$, by Theorem 12.1, leading thereby to a Γ_m-problem $F\mathbf{P}$ which can then be solved by the methods of Ref. 5.

In § 9 and § 10 we define mappings useful in proving in § 11 that an arbitrary reduced Γ_m-problem is equivalent to a uniform Γ_m-problem.

The effect of transformations of reduced Γ_m-problems $\mathbf{P}$, on the accessory mappings of $\mathbf{P}$, is important for our program.

Lemma 8.4. Let $\mathbf{P}=(\Phi, L, \Gamma_m)$ be a reduced Γ_m-problem on which a mapping f operates externally, and a reduced mapping F of $E\times\Gamma_m$ onto $E\times\Gamma_m$ operates internally.

(i) A mapping $p\to\vartheta_1(p)$, directly accessory to $\mathbf{P}$, is directly accessory to $\mathbf{P}F$.

(ii) A mapping $p\to\vartheta_3(p)$, radially accessory to $\mathbf{P}$, is radially accessory to $f\mathbf{P}$.

(iii) A mapping $p\to\vartheta_3(p)$, radially accessory to $\mathbf{P}$, is radially accessory to $\mathbf{P}F$, provided

$$s_\rho \supset F^p(s_\rho) \qquad (p\in\Gamma_m) \tag{8.6}$$

for each shell s_ρ for which $0<\rho<1$.

Proof of (i). By definition of $\vartheta_1(p)$

$$B(2\vartheta_1(p)) \subset \Phi^p(L^p) \tag{8.7}$$

for each $p\in\Gamma_m$. By virtue of (2.18) and (2.19) the right member of (8.7) equals $(\Phi F)^p(F^{-1}(L))^p$, so that the mapping $p\to\vartheta_1(p)$ is directly accessory to the problem $\mathbf{P}F$.

Proof of (ii). The mapping $p\to\vartheta_3(p)$ is radially accessory to $\mathbf{P}$ if $0<\vartheta_3(p)<1$ and

$$s_r \subset L^p \qquad (r=\vartheta_3(p)) \tag{8.8}$$

for each $p\in\Gamma_m$. Now $f\mathbf{P}=(f\Phi, L, \Gamma_m)$, so that L is associated as a neighborhood of $S\times\Gamma_m$ both with $\mathbf{P}$ and $f\mathbf{P}$. The condition (8.8) takes the same form accordingly for $f\mathbf{P}$ as for $\mathbf{P}$ so that (ii) holds.

Proof of (iii). If (8.6) holds, as well as (8.8), then for $r=\vartheta_3(p)$, and arbitrary $p\in\Gamma_m$,

$$s_r \subset (F^p)^{-1}(s_r) \subset (F^{-1}(L))^p \tag{8.9}$$

so that the mapping $p\to\vartheta_3(p)$ is radially accessory to $\mathbf{P}F$.

§9. The family of mappings $\mathbf{g}^a$.

Each mapping $\mathbf{g}^a$ of the family of reduced C^∞-diffeomorphisms of E onto E, here to be defined, will leave each $(n-1)$-sphere concentric with S invariant as a set. In contradistinction each mapping $\mathbf{k}^c$ of the family of reduced C^∞-diffeomorphisms to be defined in §10 will leave each ray

emanating from the origin invariant as a set. Together these mappings will enable us to establish the existence of a uniform Γ_m-problem equivalent to a given reduced Γ_m-problem $\mathbf{P}$.

Both in § 9 and in § 10 we shall have occasion to apply the following lemma.

Lemma 9.0. *Let a C^∞-mapping $\mathbf{x} \to f(\mathbf{x})$ of E into E be such that the origin goes into the origin, and that for each positive number t, the $(n-1)$-sphere S_t on which $\|\mathbf{x}\| = t$, suffers a C^∞-diffeomorphism onto an $(n-1)$-sphere $S_{t'}$ on which $\|\mathbf{x}\| = t' > 0$, where the mapping $t \to t'$ is a C^∞-diffeomorphism of the interval $(0, +\infty)$ onto the interval $(0, +\infty)$. If moreover the jacobian*

$$D(\mathbf{x}) = \frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)}(\mathbf{x})$$

does not vanish at the origin, the mapping $\mathbf{x} \to f(\mathbf{x})$ is a C^∞-diffeomorphism of E onto E.

That the mapping $\mathbf{x} \to f(\mathbf{x})$ is biunique and onto E is immediate. It remains to show that $D(\mathbf{x})$ does not vanish when $\|\mathbf{x}\| > 0$.

Let $\mathbf{a}$ be an arbitrary point of E such that $\|\mathbf{a}\| = t > 0$. Let $(V_1, \ldots, V_n)$ be independent contravariant vectors at $\mathbf{a}$ such that the vectors $V_1, \ldots, V_{n-1}$ are tangent to S_t at $\mathbf{a}$, and V_n is orthogonal to S_t at $\mathbf{a}$. Set $f(\mathbf{a}) = \mathbf{a}'$ and $t' = \|\mathbf{a}'\|$. Under the transformation of contravariant vectors associated with the given mapping at $\mathbf{a}$, the vectors (V) go into a set of vectors $(V'_1, \ldots, V'_n)$ at $\mathbf{a}'$ of which $(V'_1, \ldots, V'_{n-1})$ are independent and tangent at $\mathbf{a}'$ to the $(n-1)$-sphere $S_{t'}$, on which $\|\mathbf{x}\| = t'$. Moreover V'_n is not null nor tangent to $S_{t'}$ at $\mathbf{a}'$ since $dt'/dt \neq 0$ by hypothesis. Hence the vectors $(V'_1, \ldots, V'_n)$ are independent. We infer that $D(\mathbf{a}) \neq 0$.

This establishes Lemma 9.0.

Let a be a constant on the open interval (0,2). Set

$$\eta_a = S \cap B(a). \tag{9.0}$$

The lemma characterizing the family $\mathbf{g}^a$ is as follows.

Lemma 9.1. *There exists a C^∞-mapping*

$$\mathbf{g} : E \times (0, 2) \to E, \quad (\mathbf{x}, a) \to \mathbf{g}(\mathbf{x}, a) \tag{9.1}$$

such that the partial mappings $\mathbf{g}^a : \mathbf{x} \to \mathbf{g}(\mathbf{x}, a)$, $0 < a < 2$, have the following properties.

(a.1) The mapping $\mathbf{g}^a$ is a reduced C^∞-diffeomorphism of E onto E.

(a.2) Each $(n-1)$-sphere with center at the origin is invariant as a set under $\mathbf{g}^a$.

(a.3) The image of η_a under $\mathbf{g}^a$ is η_1.

(a.4) For $\mathbf{x} \in E$, $\mathbf{g}(\mathbf{x}, 1) = \mathbf{x}$. Moreover

$$\mathbf{g}^a(\mathbf{x}) = \|\mathbf{x}\| \, \mathbf{g}^a\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) \qquad \left(\|\mathbf{x}\| > \frac{1}{2},\ 0 < a < 2\right). \tag{9.2}$$

The mapping $\mathbf{g}$, when restricted to $S \times (0, 2)$ will be identified with the mapping $\mathbf{h}$ defined in Lemma 9.2.

Lemma 9.2. There exists a real analytic mapping

$$\mathbf{h} : S \times (0, 2) \to S, \quad (\mathbf{x}, a) \to \mathbf{h}(\mathbf{x}, a) \tag{9.3}$$

such that for fixed $a \in (0, 2)$, $\mathbf{h}^a$ is a real non-singular, analytic diffeomorphism of S onto S that leaves Q fixed, maps η_a onto η_1 and reduces to the identity for $a = 1$.

The reflection H. Let H be a reflection of $E - Q$ in $\beta B(1)$, the boundary of $B(1)$. Let $\gamma\eta_a$ be the boundary of η_a, relative to S, and note that

$$\gamma\eta_a = S \cap \beta B(a) \qquad (0 < a < 2). \tag{9.4}$$

Let π be the $(n-1)$-plane through $\gamma\eta_1$. Let $\mathbf{e}$ be the point of intersection of π and the x_n-axis. The image of $S - Q$ under H is π.

The radial mapping G_a. For $0 < a < 2$ let G_a be a radial mapping of E onto E with center $\mathbf{e}$, in which $\mathbf{y} \in E$ goes into $\mathbf{z} \in E$ in such a manner that

$$\mathbf{z} - \mathbf{e} = \varkappa(a)(\mathbf{y} - \mathbf{e}) \qquad (\varkappa(a) > 0). \tag{9.5}$$

We shall presently determine $\varkappa(a)$ by the condition that under the mapping

$$\mathbf{x} \to (G_a H)(\mathbf{x}) \tag{9.6}$$

$\gamma\eta_a$ is mapped onto $\gamma\eta_1$. Recall that $\mathbf{q}$ represents Q.

Definition of $\mathbf{h}$. For each $a \in (0, 2)$ set $\mathbf{h}(\mathbf{q}, a) = \mathbf{q}$ and

$$\mathbf{h}(\mathbf{x}, a) = (H^{-1} G_a H)(\mathbf{x}) \qquad (\mathbf{x} \in S - \mathbf{q}). \tag{9.7}$$

It is clear that $\mathbf{h}^a$ maps S onto S for $a \in (0, 2)$.

We shall now show that

$$\varkappa^2(a) = \frac{3a^2}{4 - a^2} \qquad (0 < a < 2). \tag{9.8}$$

Observe first that under H a point $\mathbf{x} \in E - \mathbf{q}$ goes into a point $\mathbf{y} \in E$ such that

$$\mathbf{y} - \mathbf{q} = \frac{\mathbf{x} - \mathbf{q}}{\|\mathbf{x} - \mathbf{q}\|^2} \qquad (\mathbf{x} \neq \mathbf{q}) \tag{9.9}$$

and that H has the inverse

$$\mathbf{x} - \mathbf{q} = \frac{\mathbf{y} - \mathbf{q}}{\|\mathbf{y} - \mathbf{q}\|^2} \qquad (\mathbf{y} \neq \mathbf{q}). \tag{9.10}$$

Let $\mathbf{x}^*$ be a point of $\gamma\eta_a$ and set

$$\mathbf{y}^* = H(\mathbf{x}^*), \quad \mathbf{z}^* = G_a(\mathbf{y}^*). \tag{9.11}$$

Since $\|\mathbf{x}^* - \mathbf{q}\| = a$ it follows from (9.9) that $\|\mathbf{y}^* - \mathbf{q}\| = 1/a$. Now $\mathbf{y}^*$ and $\mathbf{z}^*$ are in π. By hypothesis on $\varkappa(a)$, $\mathbf{z}^*$ is in $S \cap \pi$. Taking account of the fact that $\mathbf{e}_n = \mathbf{y}^*_n = \mathbf{z}^*_n = \frac{1}{2}$ and $\mathbf{q}_n = 1$, one finds that

$$\frac{1}{a^2} = \|\mathbf{y}^* - \mathbf{q}\|^2 = \|\mathbf{y}^* - \mathbf{e}\|^2 + \frac{1}{4} \tag{9.12}$$

$$1 = \|\mathbf{z}^* - \mathbf{q}\|^2 = \|\mathbf{z}^* - \mathbf{e}\|^2 + \frac{1}{4} \tag{9.13}$$

$$\|\mathbf{z}^* - \mathbf{e}\|^2 = \varkappa^2(a)\,\|\mathbf{y}^* - \mathbf{e}\|^2 \qquad \text{[Cf. (9.5)]} \tag{9.14}$$

so that

$$1 - \frac{1}{4} = \varkappa^2(a)\left(\frac{1}{a^2} - \frac{1}{4}\right),$$

implying (9.8).

Observe that $\varkappa(1) = 1$ so that $\mathbf{h}(\mathbf{x}, 1) = \mathbf{x}$ for $\mathbf{x} \in S$.

The non-singular, analytic character of $\mathbf{h}$ on $S \times (0, 2)$ follows from (9.8), and from general principles in the geometry of inversion, but may be directly verified by the reader on writing out the components

$h_1(\mathbf{x}, a), \ldots, h_n(\mathbf{x}, a)$ of $\mathbf{h}(\mathbf{x}, a)$ as rational functions of $(x_1, \ldots, x_n)$ and a.

It is clear that each $\mathbf{h}^a$ maps η_a onto η_1.

This establishes Lemma 9.2.

Proof of Lemma 9 1. We shall make use of a C^∞-mapping μ of the real axis R onto $[0, 1]$ such that $\mu(t) = 1$ for $t \geqq \frac{1}{2}$, and $\mu(t) = 0$ for $t \leqq \frac{1}{4}$. For $t \in R$ and $0 < a < 2$ set

$$A(t, a) = \mu(t) a + 1 - \mu(t). \tag{9.15}$$

Note that $A(t, 1) = 1$. The mapping

$$A : R \times (0, 2) \to R; \quad (t, a) \to A(t, a) \tag{9.16}$$

is of class C^∞, such that $0 < A(t, a) < 2$ and

$$A(t, a) = a \qquad (t \geqq \tfrac{1}{2}) \tag{9.17$'$}$$

$$A(t, a) = 1 \qquad (t \leqq \tfrac{1}{4}). \tag{9.17$''$}$$

With the aid of A one obtains, for fixed t and a, a C^∞-diffeomorphism

$$\mathbf{x} \to \mathbf{h}(\mathbf{x}, A(t, a)) \qquad (t \in R,\ 0 < a < 2) \tag{9.18}$$

of S onto S, leaving Q fixed.

For each $a \in (0, 2)$ and $\mathbf{x} \in E$ such that $\|\mathbf{x}\| \neq 0$, set

$$\mathbf{g}(\mathbf{x}, a) = \|\mathbf{x}\| \mathbf{h}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}, A(\|\mathbf{x}\|, a)\right) \tag{9.19}$$

and complete the definition of $\mathbf{g}$ by setting $\mathbf{g}(\mathbf{O}, a) = \mathbf{O}$.

In accord with (9.17)

$$\mathbf{g}(\mathbf{x}, a) = \mathbf{x} \qquad (0 \leqq \|\mathbf{x}\| \leqq \tfrac{1}{4}) \tag{9.20}$$

$$\mathbf{g}(\mathbf{x}, a) = \|\mathbf{x}\| \mathbf{h}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}, a\right) \qquad (\|\mathbf{x}\| \geqq \tfrac{1}{2}). \tag{9.21}$$

In particular (9.21) implies that on S

$$\mathbf{g}(\mathbf{x}, a) = \mathbf{h}(\mathbf{x}, a). \tag{9.22}$$

The mapping $\mathbf{g}$ so defined satisfies Lemma 9.1.

Proof of (a.1) and (a.2). Each $\mathbf{g}^a$ reduces to the identity for $\|\mathbf{x}\| < \frac{1}{4}$, and for every value of $t > 0$ the $(n-1)$-sphere on which $\|\mathbf{x}\| = t$ undergoes a C^∞-diffeomorphism onto itself in accord with (9.19). It follows from Lemma 9.0 that each $\mathbf{g}^a$ is a C^∞-diffeomorphism of E onto E. The mapping $t \to t'$ referred to in Lemma 9.0 is here the identity.

Proof of (a.3). This statement follows from (9.22) and the fact that under $\mathbf{h}^a$, as defined by $\overline{G_a H}$, followed by H^{-1}, the restricted mapping of $\gamma\eta_a$ onto $\gamma\eta_1$ is followed by the identity mapping of $\gamma\eta_1$ onto $\gamma\eta_1$. The mapping of η_a onto η_1 is thereby determined.

Proof of (a.4). That $\mathbf{g}(\mathbf{x}, 1) = \mathbf{x}$ on E follows from (9.19) and the identity $\mathbf{h}(\mathbf{x}, 1) = \mathbf{x}$ on S. Finally (9.2) follows from (9.21) and (9.22).

§ 10. The family of mappings $\mathbf{k}^c$.

We shall refer to the shell s_ρ, defined in (7.8), and prove Lemma 10.1. The parameter c ranges on the interval $(1, 2)$.

Lemma 10.1. There exists a C^∞-mapping

$$\mathbf{k} : E \times (1, 2) \to E \; ; \; (\mathbf{x}, c) \to \mathbf{k}(\mathbf{x}, c) \tag{10.1}$$

such that the partial mapping $\mathbf{k}^c$ has the following properties for each $c \in (1, 2)$.

(b.1) $\mathbf{k}^c$ is a reduced C^∞-diffeomorphism of E onto E in which each $(n-1)$-sphere with center at $\mathbf{O}$ undergoes a C^∞-diffeomorphism onto an $(n-1)$-sphere with center at $\mathbf{O}$.

(b.2) Under $\mathbf{k}^c$ each ray emanating from $\mathbf{O}$ is mapped onto itself.

(b.3) For $0 < \rho < 1$, the shell s_ρ is mapped by $\mathbf{k}^c$ onto s_1 when $c = 1 + \rho$, and for arbitrary $c \in (1, 2)$ is mapped by $\mathbf{k}^c$ onto a shell which includes s_ρ.

The mapping α. To define $\mathbf{k}$, for each $c \in (1, 2)$ we shall introduce a mapping $t \to \alpha(t, c)$ of the interval $-1 < t < \infty$ into the positive real axis R^+, by setting

$$\alpha(t, c) = \operatorname{Exp}\left[\tau(c)\left(\frac{t-1}{t+1}\right)\right], \tag{10.2}$$

where $\tau(c)$ is a positive constant to be determined for each value of $c \in (1, 2)$ by the condition

$$c\,\alpha(c^2, c) = 2\,. \tag{10.3}$$

Taking the logarithms of the members of (10.3) one has the condition

$$\log c + \tau(c)\left(\frac{c^2-1}{c^2+1}\right) = \log 2 \tag{10.4}$$

implying that

$$\tau(c) = \frac{c^2+1}{c^2-1}\log\frac{2}{c} > 0 \qquad (1<c<2). \tag{10.5}$$

The mapping $c \to \tau(c)$ is thus real, positive and analytic for $1<c<2$.

The mapping $(t, c) \to \alpha(t, c)$ into R^+ is of class C^∞ and such that

$$\alpha(1, c) = 1 \qquad \frac{\partial}{\partial t}\alpha(t^2, c) > 0 \qquad (t>0,\ 1<c<2) \tag{10.6}$$

$$\alpha\left(\frac{1}{u}, c\right) = \frac{1}{\alpha(u, c)} \qquad (u>0,\ 1<c<2) \tag{10.7}$$

$$c\,\alpha(c^2, c) = 2, \quad \frac{1}{c}\alpha\left(\frac{1}{c^2}, c\right) = \frac{1}{2} \qquad (1<c<2) \tag{10.8}$$

$$\frac{\partial}{\partial t} t\,\alpha(t^2, c) > 1 \qquad (t>1,\ 1<c<2). \tag{10.9}$$

These relations are readily verified.

The mapping $\mathbf{k}$. Let $\mathbf{k}$ be defined by setting

$$\mathbf{k}(\mathbf{x}, c) = \mathbf{x}\alpha(\|\mathbf{x}\|^2, c) \qquad (\mathbf{x}\in E,\ 1<c<2). \tag{10.10}$$

Proof of Lemma 10.1. It is clear that the mapping $(\mathbf{x}, c) \to \mathbf{k}(\mathbf{x}, c)$ is of class C^∞. We shall accordingly verify (b.1), (b.2) and (b.3).

Proof of (b.1). By (10.6), $\alpha(1, c) = 1$ for $1<c<2$, so that $\mathbf{k}^c$ reduces to the identity on S. By definition $\mathbf{k}(\mathbf{O}, c) = \mathbf{O}$. Thus each mapping $\mathbf{k}^c$ is reduced.

For any fixed positive constant t, the $(n-1)$-sphere on which $\|\mathbf{x}\| = t$ suffers a C^∞-diffeomorphism onto the $(n-1)$-sphere on which $\|\mathbf{x}\| = t'$, where

$$t' = t\,\alpha(t^2, c) \qquad (t>0). \tag{10.11}$$

For $t>0$ this mapping $t \to t'$ is a C^∞-diffeomorphism of the positive t-axis onto the positive t'-axis. This is a consequence of the differential conditions in (10.6) and (10.9). It follows then that each mapping $\mathbf{k}^c$ is onto E.

It remains to show that each $\mathbf{k}^c$ is a C^∞-diffeomorphism. To that end observe first that the jacobian

$$\frac{D(\mathbf{k}_1^c, \ldots, \mathbf{k}_n^c)}{D(x_1, \ldots, x_n)} \tag{10.12}$$

does not vanish at the origin. That $\mathbf{k}^c$ is a C^∞-diffeomorphism of E onto E now follows from Lemma 9.0.

Proof of (b.2). That a ray emanating from $\mathbf{O}$ is mapped by $\mathbf{k}^c$ into itself, follows at once from (10.10). That this mapping is onto, follows from the fact that the mapping $t \rightarrow t'$ defined by (10.11) is a mapping of the positive t-axis onto the positive t'-axis.

Proof of (b.3). The relations (10.8) imply that the $(n-1)$-spheres on which $\|\mathbf{x}\| = c$ and $\frac{1}{c}$ respectively are carried by $\mathbf{k}^c$ into the $(n-1)$-spheres on which $\|\mathbf{x}\| = 2$ and $\|\mathbf{x}\| = \frac{1}{2}$ respectively. For $0 < \rho < 1$ the shell s_ρ is accordingly mapped by $\mathbf{k}^{1+\rho}$ onto the shell s_1. Cf. (7.8).

Consider the mapping of a shell s_ρ, $0 < \rho < 1$, under $\mathbf{k}^c$, $1 < c < 2$. Set $t = 1 + \rho$. Under the mapping $\mathbf{k}^c$ the $(n-1)$-spheres on which $\|\mathbf{x}\| = t$ and $\frac{1}{t}$ respectively go into $(n-1)$-spheres with radii

$$t' = t\,\alpha(t^2, c) \quad , \quad t'' = \frac{1}{t}\,\alpha\left(\frac{1}{t^2}, c\right)$$

respectively, that is, with radii t' and $\frac{1}{t'}$ by virtue of (10.7). Thus the image of s_ρ under $\mathbf{k}^c$ is a shell $s_{\rho'}$ where $1 + \rho' = t'$. Since $t > 1$ and (10.9) holds, $t' > t$, and hence $\rho' > \rho$. Thus $s_{\rho'} \supset s_\rho$ and the proof of (b.3) is complete.

§ 11. The fundamental equivalence theorem.

A uniform Γ_m-problem was defined in § 8. In this section we shall show that an arbitrary Γ_m-problem is equivalent to a uniform Γ_m-problem. An arbitrary Γ_m-problem is externally equivalent to a reduced Γ_m-problem (Lemma 6.1). We continue with such a problem.

Lemma 11.1. *An arbitrary reduced Γ_m-problem is externally equivalent to a reduced Γ_m-problem to which a constant mapping $p \rightarrow c_1$ is directly accessory.*

Let $\mathbf{P} = (\Phi, L, \Gamma_m)$ be the given reduced Γ_m-problem. To establish the lemma it suffices to prove the following. There exists a positive constant c_1 and a p-invariant C^m-diffeomorphism F of $E \times \Gamma_m$ into $E \times \Gamma_m$ such that

$$B(2c_1) \subset (F\Phi)^p (L^p) \qquad (p \in \Gamma_m). \tag{11.1}$$

To define F let $p \rightarrow \vartheta_1(p)$ be a C^m-mapping of Γ_m into R^+, directly

accessory to **P** (Lemma 8.2). By virtue of Lemmas 1.1 and 1.2 one can define a p-invariant C^m-diffeomorphism of $E \times \Gamma_m$ onto $E \times \Gamma_m$ by the conditions

$$F^p(\mathbf{x}) - \mathbf{q} = \frac{\mathbf{x} - \mathbf{q}}{\vartheta_1(p)} \qquad (\mathbf{x} \in E,\ p \in \Gamma_m). \tag{11.2}$$

The definition of $\vartheta_1(p)$ as a constant directly accessory to the simple reduced problem $\mathbf{P}^p$, implies that

$$B(2\vartheta_1(p)) \subset \Phi^p(L^p). \tag{11.3}$$

We shall apply the mapping F^p to both members of (11.3), obtaining thereby the relation (Cf. Lemma 2.5)

$$B(2) \subset F^p(\Phi^p(L^p)) = (F\Phi)^p(L^p). \tag{11.4}$$

Thus the mapping $p \to 1$ is directly accessory to $F\mathbf{P}$. Cf. § 7.

This establishes Lemma 11.1.

Lemma 11.2. *An arbitrary reduced Γ_m-problem* **P** *is internally equivalent to a reduced Γ_m-problem* $\mathbf{P}^*$ *to which the constant mapping $p \to 1/2$ is radially accessory. Any mapping $p \to \vartheta_1(p)$ directly accessory to* **P** *is directly accessory to* $\mathbf{P}^*$ (Lemma 8.4 (i)).

Set $\mathbf{P} = (\Phi, L, \Gamma_m)$. If f is a reduced C^m-diffeomorphism of $E \times \Gamma_m$ onto $E \times \Gamma_m$, the problem $\mathbf{P}f$ has the form $(\Phi f, f^{-1}(L), \Gamma_m)$ by definition, and its p-sections have the form

$$(\mathbf{P}f)^p = ((\Phi f)^p, (f^{-1}(L))^p, \Gamma_m).$$

We shall choose f so that the constant mapping $p \to 1/2$ is radially accessory to the Γ_m-problem $\mathbf{P}f$. For this it is sufficient that the shell $s_{1/2}$ admit the inclusion

$$s_{1/2} \subset (f^{-1}(L))^p \tag{11.5}$$

for each $p \in \Gamma_m$. Cf. (7.9).

Definition of f. To define f, let $p \to \vartheta_3(p)$ be a C^m-mapping radially accessory to **P** (Lemma 8.3). We refer to the mapping **k** introduced in Lemma 10.1, and let f be the p-invariant homeomorphism f of $E \times \Gamma_m$ onto $E \times \Gamma_m$ determined by the conditions

$$(f^p)^{-1} = \mathbf{k}^{1+\vartheta_3(p)} \qquad (p \in \Gamma_m) \tag{11.6}$$

recalling that $0 < \vartheta_3(p) < 1$. Cf. (7.9). The mapping f exists by Lemma 1.1. Because of the properties of $\mathbf{k}$, f is a reduced mapping of $E \times \Gamma_m$ onto $E \times \Gamma_m$. That is, both $(f^p)^{-1}$ and f^p leave the origin and $\mathbf{q}$ fixed and leave S invariant as a set. Moreover f is of class C^m over $E \times \Gamma_m$, while f^p is a C^m-diffeomorphism of E onto E for each $p \in \Gamma_m$, in accord with the properties of $\mathbf{k}^c$ listed in Lemma 10.1 and the choice of $\vartheta_3(p)$. Hence f is a C^m-diffeomorphism of $E \times \Gamma_m$ onto $E \times \Gamma_m$. Lemma 1.2.

According to the definition of $\vartheta_3(p)$, (Cf. (7.9)), $s_\rho \subset L^p$ for $\rho = \vartheta_3(p)$ and for $p \in \Gamma_m$. It follows from (b.3) of Lemma 10.1, and from (11.6), that

$$s_1 = (f^p)^{-1}(s_\rho) \qquad (\text{for } \rho = \vartheta_3(p)). \tag{11.7}$$

Hence

$$s_1 = (f^p)^{-1}(s_\rho) \subset (f^{-1})^p(L^p) = (f^{-1}(L))^p \tag{11.8}$$

thereby implying (11.5).

In proving Lemma 11.3 below we shall make use of the mappings $\mathbf{g}^a$ introduced in Lemma 9.1. For this purpose neighborhoods of Q of a special type naturally enter.

Neighborhoods $R(\beta, \gamma)$ of Q. Recall the sets $\eta_a = S \cap B(a)$, $0 < a < 2$, introduced in § 9. For $0 < 2\beta \leqq 1$ and $0 < \gamma \leqq 1$, with β and γ fixed, set

$$R(\beta, \gamma) = \left\{ \mathbf{x} \mid \mathbf{x} = t\mathbf{y} \,,\ \mathbf{y} \in \eta_\beta \,,\ \frac{1}{1+\gamma} < t < 1 + \gamma \right\}. \tag{11.9}$$

The set $R(\beta, \gamma)$ is an open neighborhood of Q. It may be described as the intersection of the half cone of points on rays joining the origin Z to η_β, with the region between the $(n-1)$-spheres on which $\|\mathbf{x}\| = 1 + \gamma$ and $1/1 + \gamma$ respectively. We shall prove statement I.

I. If $0 < 2\rho \leqq 1$, $R(\rho, \rho) \subset B(2\rho)$.

Let $\mathbf{a}_\rho$ be the point on the x_n-axis at which $x_n = 1 + \rho$. The subset of points of $Cl\, R(\rho, \rho)$ at the maximum distance from Z is the intersection X_ρ of the $(n-1)$-sphere

$$\{\mathbf{x} \mid \|\mathbf{x}\| = 1 + \rho\} \tag{11.10}$$

with the closed n-ball

$$\{\mathbf{x} \mid \|\mathbf{x}-\mathbf{a}_\rho\| \leqq \rho(1+\rho)\}. \tag{11.11}$$

It is clear that I holds if $X_\rho \subset B(2\rho)$, and that this inclusion holds if an arbitrary point $\mathbf{y} \in E$ at which

$$\|\mathbf{y}\| = 1+\rho, \quad \|\mathbf{y}-\mathbf{a}_\rho\| = \rho(1+\rho) \tag{11.12}$$

is in $Cl\, B(2\rho)$. The point $\mathbf{y}$ projects orthogonally into a point $\mathbf{x} \in S$ such that $\|\mathbf{y}-\mathbf{x}\| = \rho$ and $\|\mathbf{x}-\mathbf{q}_\rho\| = \rho$. Hence

$$\|\mathbf{y}-\mathbf{q}\| \leqq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{x}-\mathbf{q}\| = 2\rho. \tag{11.13}$$

Thus $\mathbf{y}$ is in $Cl\, B(2\rho)$ and $X_\rho \subset B(2\rho)$, implying I.

We continue by proving II.

II. *If* $\mathbf{g}$ *and* $\mathbf{k}$ *are the mappings introduced in Lemmas* 9.1 *and* 10.1, *respectively, then*

$$(\mathbf{k}^{1+\rho}\mathbf{g}^\rho)(R(\rho, \rho)) = R(1, 1) \qquad (0 < 2\rho \leqq 1). \tag{11.14}$$

The relation

$$\mathbf{g}^\rho(R(\rho, \rho)) = R(1, \rho) \tag{11.15}$$

is a consequence of (a.3) and (a.4) of Lemma 9.1. In this application of (a.4), we note that the condition $\|\mathbf{x}\| > \frac{1}{2}$ is satisfied for points $\mathbf{x} \in R(\rho, \rho)$, since for such $\mathbf{x}$

$$\|\mathbf{x}\| > \frac{1}{1+\rho} > \frac{1}{2}.$$

We continue by proving that

$$\mathbf{k}^{1+\rho}(R(1, \rho)) = R(1, 1). \tag{11.16}$$

The range of $\|\mathbf{x}\|$ on $R(1, \rho)$ is exactly that on the shell s_ρ of Lemma 10.1. Moreover by Lemma 10.1

$$\mathbf{k}^{1+\rho}(s_\rho) = s_1.$$

The range of $\|\mathbf{x}\|$ on $R(1, 1)$ is that on s_1. With this understood, (11.16) follows readily on recalling that the mappings $\mathbf{k}^c$ leave rays emanating from the origin invariant as sets.

Equations (11.15) and (11.16) together imply (11.14), completing the proof of II.

By virtue of Lemmas 11.1 and 11.2 an arbitrary reduced Γ_m-problem

is equivalent to a reduced Γ_m-problem to which constant mappings $p \to c_1$ and $p \to c_3$ of Γ_m into R^+ are directly and radially accessory respectively. We are led thereby to a final lemma.

Lemma 11.3. *An arbitrary reduced Γ_m-problem*

$$\mathbf{P} = (\Phi , L , \Gamma_m)$$

to which constant mappings $p \to c_1$ and $p \to c_3$ are directly and radially accessory respectively, is internally equivalent to a uniform Γ_m-problem $\mathbf{P}F$ *to which the constant mappings $p \to c_1$ and $p \to c_3$ are again directly and radially accessory, and to which a constant mapping $p \to c_2$ is inversely accessory with*

$$(\Phi F)^p (B(c_2)) \subset B(c_1) \qquad (p \in \Gamma_m) . \tag{11.17}$$

In accord with Lemma 8.3 there exists a C^m-mapping $p \to \vartheta_2(p)$ of Γ_m into R^+, such that

$$\Phi^p (B(\vartheta_2(p))) \subset B(c_1) \qquad (p \in \Gamma_m) . \tag{11.18}$$

To prove Lemma 11.3 it suffices to prove that there exists a reduced mapping F of $E \times \Gamma_m$ onto $E \times \Gamma_m$ satisfying (8.6) and a positive constant c_2 such that (11.17) holds, with $0 < 2c_2 \leq 1$. See Lemma 8.4.

Definition of F. We shall define F by defining its inverse f as a p-invariant reduced mapping of $E \times \Gamma_m$ onto $E \times \Gamma_m$ such that

$$f^p = \mathbf{k}^{1+r} \mathbf{g}^r \qquad (2r = \vartheta_2(p) , p \in \Gamma_m) . \tag{11.19}$$

In accord with Lemmas 1.1 and 1.2 and the properties of $\mathbf{g}$ and $\mathbf{k}$, enunciated in Lemmas 9.1 and 10.1, f as well as its inverse F, is a reduced mapping of $E \times \Gamma_m$ onto $E \times \Gamma_m$.

Choice of c_2. To simplify the notation set $B(\vartheta_2(p)) = B^p$. By virtue of I of this section, if one sets $2\rho = \vartheta_2(p)$, then

$$R(\rho , \rho) \subset B^p \qquad (p \in \Gamma_m) \tag{11.20}$$

so that the definition of f and statement II imply that

$$R(1 , 1) \subset f^p (B^p) . \tag{11.21}$$

Since F is the inverse of f the given inclusion (11.18) can be written in the form

$$(\Phi F)^p(f^p(B^p)) \subset B(c_1). \tag{11.22}$$

With the aid of (11.21), (11.22) implies that

$$(\Phi F)^p(R(1, 1)) \subset B(c_1) \qquad (p \in \Gamma_m). \tag{11.23}$$

For $\mathbf{x} \in R(1, 1)$, $\|\mathbf{x}\|$ ranges over the interval $(\frac{1}{2}, 2)$. Moreover $R(1, 1)$ includes the subset $\eta_1 = S \cap B(1)$ of S. It follows that $R(1, 1)$ includes $B(\frac{1}{2})$, and from (11.23) we conclude then that

$$(\Phi F)^p(B(\tfrac{1}{2})) \subset B(c_1). \tag{11.24}$$

Thus (11.17) holds with $c_2 = 1/2$.

It remains to prove III.

III. *With F chosen as above the mappings $p \to c_1$ and $p \to c_3$ are directly and radially accessory respectively to* $\mathbf{P}F$.

The constant mapping $p \to c_3$, given as radially accessory to $\mathbf{P}$, is also radially accessory to $\mathbf{P}F$, if the condition (8.6) of Lemma 8.4 is satisfied for each $p \in \Gamma_m$. This condition takes the form

$$s_\rho \subset f^p(s_\rho) \qquad (0 < \rho < 1) \tag{11.25}$$

on recalling that f is the inverse of F. Making use of the definition of f in (11.19), condition (11.25) takes the form

$$s_\rho \subset (\mathbf{k}^{1+r}\,\mathbf{g}^r)(s_\rho) \qquad (0 < \rho < 1) \tag{11.26}$$

with $2r = \vartheta_2(p)$. In accord with (a.2) of Lemma 9.1

$$s_\rho = \mathbf{g}^r(s_\rho) \qquad (0 < \rho < 1). \tag{11.27}$$

By virtue of Lemma 10.1 (b.3)

$$s_\rho \subset \mathbf{k}^{1+r}(s_\rho) \qquad (0 < r < 1,\ 0 < \rho < 1). \tag{11.28}$$

We conclude that (11.26) and hence (11.25) is satisfied.

The mapping $p \to c_1$ is directly accessory to $\mathbf{P}F$ by Lemma 8.4 (i).

The problem $\mathbf{P}F$ is thus a uniform Γ_m-problem which satisfies Lemma 11.3.

Taken together Lemmas 11.1, 11.2 and 11.3 imply the basic theorem of Part I.

THEOREM 11.1. *An arbitrary Γ_m-problem is equivalent to a uniform Γ_m-problem.*

PART II

PROOF OF THEOREM 1.2

§ 12. Elementary Γ_m-problems.

Theorem 7.1 has the following generalization.

Theorem 12.1. Let $\mathbf{P} = (\Phi, L, \Gamma_m)$ be an arbitrary uniform Γ_m-problem to which constant mappings $p \to c_1$, $p \to c_2$, $p \to c_3$ of Γ_m into R^+ are directly, inversely and radially accessory, respectively, with

$$(12.0) \qquad \Phi^p(B(c_2)) \subset B(c_1) \qquad (p \in \Gamma_m).$$

The Problem $\mathbf{P}$ is then externally equivalent to a Γ_m-problem $F\mathbf{P}$ such that

$$(12.1) \qquad (F\Phi)^p(\mathbf{x}) = \mathbf{x} \qquad (\mathbf{x} \in B(c_2), p \in \Gamma_m).$$

The constant mapping $p \in c_3$ remains radially accessory to $F\mathbf{P}$.

The proof of Theorem 12.1 is similar to that of Theorem 7.1. In terms of the above constant c_1, let mappings λ and μ be defined as in § 7. Let $\mathbf{M}$ be the p-invariant C^∞-diffeomorphism of $E \times \Gamma_m$ onto $B(2c_1) \times \Gamma_m$ such that $\mathbf{M}^p(\mathbf{y}) = \mu(\mathbf{y})$ for $\mathbf{y} \in E$, $p \in \Gamma_m$.

The external p-invariant operator F. Set

$$(12.2) \qquad F(\mathbf{y}, p) = \Phi^{-1}(\mathbf{M}(\mathbf{y}, p)) \qquad ((\mathbf{y}, p) \in E \times \Gamma_m).$$

The mapping F is well-defined since $\mathbf{M}(E \times \Gamma_m) = B(2c_1) \times \Gamma_m$, a set on which Φ^{-1} is defined. Moreover F is a C^m-diffeomorphism over $E \times \Gamma_m$. Note that $F(\mathbf{y}, p) = \Phi^{-1}(\mathbf{y}, p)$ for $(\mathbf{y}, p) \in B(c_1) \times \Gamma_m$ in accord with the definition of $\mathbf{M}$. For each $p \in \Gamma_m$

$$(12.3) \qquad (F\Phi)^p(\mathbf{x}) = \mathbf{x} \qquad (\mathbf{x} \mid \Phi^p(\mathbf{x}) \in B(c_1)).$$

Now the side condition in (12.3) is satisfied whenever $\mathbf{x} \in B(c_2)$, by virtue of (12.0). Hence (12.1) holds.

That the constant mapping $p \to c_3$, radially accessory to $\mathbf{P}$, remains radially accessory to $F(\mathbf{P})$, follows from Lemma 8.4 (ii).

This completes the proof of Theorem 12.1.

Definition. A Γ_m-problem with which there are associ-

ated (1) an $(n-1)$-sphere S_Q with center Q such that

$$\Phi^p(\mathbf{x}) = \mathbf{x} \qquad (\mathbf{x} \in \mathring{J} S_Q\,,\, p \in \Gamma_m) \tag{12.4}$$

and (2) a shell $\delta = s_\rho$, $0<\rho<1$, with ρ independent of $p \in \Gamma_m$ and such that $\delta \subset L^p$ for each $p \in \Gamma_m$, will be termed elementary.

The problem $F\mathbf{P}$ of Theorem 12.1 with $\mathring{J} S_Q = B(c_2)$ and $\delta = s_{c_3}$, is elementary in the above sense. Hence Theorems 11.1 and 12.1 imply the following corollary.

Corollary 12.1. An arbitrary Γ_m-problem is equivalent to an elementary Γ_m-problem.

§ 13. The reflection t.

According to Corollary 7.1 there exists a simple elementary problem externally equivalent to an arbitrary simple Schoenflies problem. Let $P = (\varphi\,, N\,, m)$ be such a simple elementary problem. With s_ρ defined as in (7.8) let $\delta = s_\rho$ where ρ is a constant radially accessory to P. Since P is elementary there exists a sphere S_Q with center at Q such that

$$\varphi(\mathbf{x}) = \mathbf{x} \qquad (\mathbf{x} \in \mathring{J} S_Q)\,. \tag{13.1}$$

We shall suppose that $J S_Q \subset \delta$ and that the radius r of S_Q is at most 1/4. We shall make a reflection t of $E - Q$ in S_Q. In this way we shall obtain a new problem termed the t-transform of P, a problem which is, for our purposes, simpler than P. In § 15 we shall similarly define the $\mathbf{t}$-transform of an elementary Γ_m-problem $\mathbf{P}$.

We shall be as explicit as possible in our definition of auxiliary sets K, H', Π, etc., in order that it will appear that these definitions need no alteration when we turn in § 15 to an elementary Γ_m-problem, or more precisely that these sets do not depend on the particular simple problem

$$\mathbf{P}^p = (\Phi^p\,, L^p\,, m) \qquad (p \in \Gamma_m)$$

under consideration in § 13.

Under t the image of $S - Q$ is an $(n-1)$-plane Π through $S_Q \cap S$ on which $x_n = 1 - \frac{r^2}{2}$. Up to this point the origin has been Z the center of S.

For the remainder of this paper we shall suppose that the origin of coordinates has been rechosen as the intersection of the x_n-axis with Π.

The facts which unify our treatment of the simple elementary problem P and the elementary Γ_m-problem $\mathbf{P}$ to be treated in § 15, are that for a suitable choice of $\rho > 0$, independent of $p \in \Gamma_m$, the shell $\delta = s_\rho \subset L^p$, and that for a suitable choice of S_Q, independent of $p \in \Gamma_m$, $\Phi^p(\mathbf{x}) = \mathbf{x}$, for $\mathbf{x} \in \mathring{J} S_Q$. We shall suppose that $J S_Q \subset \delta$.

With this understood set

$$\psi(t(\mathbf{x})) = t\varphi(\mathbf{x}) \qquad (\mathbf{x} \in \delta - Q). \tag{13.2}$$

So defined, ψ is a C^m-diffeomorphism of the subset $t(\delta - Q)$ of E into E, such that ψ reduces to the identity $\mathbf{U}$ on $E - J S_Q$.

The rectangle K. Let K be the open n-cube of points $\mathbf{x} \in E$, such that $-1 < x_i < 1$, $i = 1, \dots, n$. Now $J S_Q \subset K$, and ψ reduces to the identity on the open complement $C J S_Q$ of $J S_Q$. We regard $C J S_Q$ as a neighborhood of the closed set CK. The open set on which ψ is defined includes the $(n-1)$-plane Π. On Π, $x_n = 0$. The set on which ψ is undefined and on which $x_n < 0$, is a closed subset of K of the form

$$\zeta' = t(JS - \delta) \tag{13.3}$$

and the set on which ψ is undefined and on which $x_n > 0$, is a closed subset of K of the form

$$\zeta'' = t(C \mathring{J} S - \delta) \cup Q \qquad (Q \in \mathring{\zeta}''). \tag{13.4}$$

The center Z of S, and $t(Z)$. If Z is the center of S, the point $t(Z)$ is a point in K on the x_n-axis with $x_n < 0$. Moreover $t(Z) \in \zeta'$.

The constant d. Let $K_{\nu d}$, $\nu = 1, 2, 3, 4$, be the open subrectangles of K on which

$$-1 + \nu d < x_i < 1 - \nu d \qquad (i = 1, \dots, n). \tag{13.5}$$

Choose $d > 0$ so small that ψ is the identity $\mathbf{U}$ on $K - K_{4d}$.

Let the open subrectangles of $K_{\nu d}$ on which $x_n < -\nu d$ and $x_n > \nu d$, respectively, be denoted by

$$H', H'' \qquad \text{(when } \nu = 1)$$
$$L', L'' \qquad \text{(when } \nu = 2)$$

and let the closures of the subrectangles of $K_{\nu d}$ on which $x_n \leqq -\nu d$ and $x_n \geqq \nu d$, respectively, be denoted by

$$G', G'' \qquad \text{(when } \nu = 3)$$
$$\Theta', \Theta'' \qquad \text{(when } \nu = 4).$$

We suppose d so small that

$$\Theta' \supset \zeta', \qquad \Theta'' \supset \zeta''. \tag{13.6}$$

We shall refer to the sets

$$H = H' \cup H'', \quad G = G' \cup G'', \quad \Theta = \Theta' \cup \Theta''.$$

Since $Z \in CN$, one has

$$t(Z) \in \zeta' \subset \Theta' \subset H'. \tag{13.7}$$

The mapping ω. Let ω be the restriction of ψ to the open set $K - \Theta$. Note that $\omega(K - \Theta) \subset K$.

The contraction $\mathbf{a}$. Let D be the open n-rectangle of points $\mathbf{x} \in E$ such that $(-1 < x_i < 9)$, $i = 1, \ldots, n$. Note that D contains the special point

$$\mathfrak{P} = (8, 0, \ldots, 0). \tag{13.8}$$

This point has been denoted by P in Ref. 5. Let $\mathbf{a}$ be a C^∞-diffeomorphism of D onto H' that leaves L' pointwise fixed. Such a contraction of D onto H' is easily set up. See Ref. 3, § 14.

The subrectangle D_0 of D. Let D_0 be a closed subrectangle of D with faces parallel to those of D, and with distances less than the above constant d from the corresponding faces of D. We see that $D_0 \supset H'$. Note that $\mathbf{a}(D_0) \supset L'$, since $D_0 \supset L'$ and L' is invariant under $\mathbf{a}$.

The simple problem (ω, H', m). The simple problem P with which we started has led us to a new problem associated with the rectangle H'. This problem will be denoted by (ω, H', m) and defined as the problem of finding a mapping λ_ω of H' into E which satisfies Theorem 13.1 as stated below. Theorem 13.1 is a special form of Theorem 2.2 of Ref. 5. It is special in the sense that the compact set Ω and point $\mathbf{w}$ which appear

in Theorem 2.2 are here replaced by $\mathbf{a}(D_0)$ and $\mathbf{a}(\mathfrak{P})$ respectively. The proof of Theorem 3.1 in Ref. 5 is essentially a proof that this special choice of Ω and $\mathbf{w}$ is permissible and satisfies Theorem 2.2 of Ref. 5 with a suitable choice of λ_ω.

Theorem 13.1. *There exists a homeomorphism λ_ω of H' onto $\mathring{J}\omega(\beta H')$ which extends $\omega | (H' - \mathbf{a}(D_0))$ and which, when $m > 0$, is in addition a C^m-diffeomorphism of $H' - \mathbf{a}(\mathfrak{P})$ into E.*

We shall call the problem (ω, H', m) a t-*transform* of the simple problem P, and shall define a $\mathbf{t}$-transform of an elementary Γ_m-problem $\mathbf{P}$ by means of an analogue of Theorem 13.1, namely Theorem 15.1. As we shall show in §§ 17, 18, Theorem 15.1 will imply Theorem 1.2, our principal theorem.

§ 14. Formulas for λ_ω.

We shall here recall the explicit formulas appearing in Ref. 5 which give a definition of a mapping λ_ω satisfying Theorem 13.1. These formulas have been derived and are here so presented as to facilitate their extension to the case of the $\mathbf{t}$-transform of an elementary Γ_m-problem $\mathbf{P}$.

Various auxiliary mappings are involved and must be described.

The radial mapping R of E onto E. Under R the point $\mathbf{x} \in E$ goes into a point $\mathbf{x}' \in E$ such that

$$x'_1 - 8 = \frac{x_1 - 8}{2}, \qquad x'_j = \frac{x_j}{2} \qquad (j = 2, \ldots, n).$$

The mapping R has $\mathfrak{P}$ as fixed point. Let R^r, $r = 1, 2, \ldots$, be the r-fold iterate of R. Let R^0 be the identity and R^{-r} the inverse of R^r.

The extension ω_e of ω. To define the domain of definition of ω_e we give E the partition,

$$E = \bigcup_{r=0}^{\infty} R^r(K) \cup A \cup \mathfrak{P}, \tag{14.0}$$

choosing A suitably. The extension ω_e of ω will be defined on $M \cup \mathfrak{P}$ where M is the open subset of E given by the disjoint union

$$M = \bigcup_{r=0}^{\infty} R^r(K - \Theta) \cup A. \tag{14.1}$$

A comparison of (14.0) and (14.1) shows that

$$CM = \bigcup_{r=0}^{\infty} R^r(\Theta) \cup \mathfrak{P}. \tag{14.2}$$

Note. For the sake of notational brevity we shall employ the following convention. When $f_1(f_2(X))$ is well-defined we shall write $f_1 \cdot f_2(X)$ for $f_1(f_2(X))$.

The definition of ω_e on the subset $R^r(K-\Theta)$ of M for $r = 0, 1, \ldots$ is given by setting

$$\omega_e(\mathbf{x}) = R^r \cdot \omega \cdot R^{-r}(\mathbf{x}) \qquad (\mathbf{x} \in R^r(K-\Theta)). \tag{14.3}$$

The definition of ω_e over M is completed by setting $\omega_e(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in A$. As stated in Lemma 5.1 of Ref. 5, ω_e, so defined, is a C^m-diffeomorphism of M into E extending $\omega \mid (K-\Theta)$. We shall need the fact that

$$\omega_e(\mathbf{x}) = \mathbf{x} \qquad (\mathbf{x} \in R^r(K-\overline{K}_d)). \tag{14.4}$$

If one sets $\omega_e(\mathfrak{P}) = \mathfrak{P}$ the mapping ω_e of $M \cup \mathfrak{P}$ into E is a homeomorphism as affirmed in § 5 of Ref. 5.

The mapping T_r. (Ref. 3, § 9 or Ref. 5, § 6.) Let B be a bounded subset of E. Let Int B denote the smallest n-rectangle π with faces parallel to the coordinate planes and such that $\pi \supset B$. There exists a C^∞-diffeomorphism T of E onto E such that

$$\boxed{\begin{array}{lll} (14.5) & T(\mathbf{x}) = \mathbf{x} & (\mathbf{x} \in K,\ x_n > d) \\ (14.6) & T(\mathbf{x}) = R(\mathbf{x}) & (\mathbf{x} \in K,\ x_n < -d) \\ (14.7) & RT(\overline{K}) \cap T(\overline{K}) = \emptyset & \\ (14.8) & T(\overline{K}) \subset \mathrm{Int}(\overline{K} \cup R\overline{K}) & \end{array}}$$

while the sign of the x_n-coordinate of $T(\mathbf{x})$ is that of the x_n-coordinate of $\mathbf{x}$. Set

$$T_{r+1} = R^r T \qquad (r = 0, 1, \ldots). \tag{14.9}$$

With the aid of (6.6) of Ref. 5 one sees that the sets

$$\mathfrak{P},\ H';\quad T_1(\overline{K}),\quad T_2(\overline{K}),\quad T_3(\overline{K}), \ldots \tag{14.10}$$

are disjoint. Since $G' \subset H'$, one accordingly has the partition

$$E = \mathfrak{P} \cup G' \cup (\bigcup_{r=1}^{\infty} T_r(K)) \cup L, \tag{14.11}$$

for appropriate choice of L. The partition is the basis of our definition of the mapping σ. We need however the following fact.

(i) *The set L in (14.11) is included in M.*

Proof of (i) Recall the partition

$$T_r(K) = T_r(K - \Theta) \cup R^r(\Theta') \cup R^{r-1}(\Theta'') \qquad (r > 0) \tag{14.12}$$

given in (6.10) of Ref. 5. To show that $L \subset M$ we make use of (14.2) and show that $CL \supset CM$. Now

$$CL = \mathfrak{P} \cup G' \cup \bigcup_{r=1}^{\infty} T_r(K)$$

in accord with (14.11). This equality, with (14.12), and the inclusion $G' \supset \Theta' = R^0(\Theta')$ shows that

$$CL \supset \mathfrak{P} \cup [\bigcup_{r=0}^{\infty} R^r(\Theta')] \cup [\bigcup_{r=0}^{\infty} R^r(\Theta'')] = CM.$$

Thus $CL \supset CM$, so that $L \subset M$.

Before coming to the definition of σ the following subsets of K require definition.

$$\boxed{\begin{array}{ll} \mathfrak{G}' = J\,\omega(\beta G') & \mathfrak{G}'' = J\,\omega(\beta G'') \\ \mathfrak{H}' = \mathring{J}\,\omega(\beta H') & \mathfrak{H}'' = \mathring{J}\,\omega(\beta H'') \\ \mathfrak{L}' = \mathring{J}\,\omega(\beta L') & \mathfrak{L}'' = \mathring{J}\,\omega(\beta L'') \\ \mathfrak{G} = \mathfrak{G}' \cup \mathfrak{G}'' & \mathfrak{H} = \mathfrak{H}' \cup \mathfrak{H}'' \end{array}} \tag{14.13}$$

The partition

$$K = \mathfrak{G} \cup \omega(K - G),$$

appearing as (5.7) of Ref. 5, is needed to show that the right member of (14.14) is well-defined.

The mapping α_r. For $r = 1, 2, \ldots$ set

$$\alpha_r(\mathbf{x}) = T_r \cdot \omega^{-1} \cdot T_r^{-1}(\mathbf{x}) \qquad (\mathbf{x} \in T_r(K - \mathfrak{G})). \tag{14.14}$$

Taking account of the fact that ω reduces to the identity over $K - \overline{K}_d$, we see that for $r = 1, 2, \ldots$

$$\alpha_r(\mathbf{x}) = \mathbf{x} \qquad (\mathbf{x} \in T_r(K - \bar{K}_d)). \tag{14.15}$$

Definition of σ over CG'. The mapping σ will be defined over each of the sets, other than G', in the partition (14.11) of E. We set $\sigma(\mathfrak{P}) = \mathfrak{P}$, and for $\mathbf{x} \in L$, set $\sigma(\mathbf{x}) = \omega_e(\mathbf{x})$, recalling that $M \supset L$.

To define σ on $T_r(K)$ use is made of the partition

$$T_r(K) + T^r(K - \mathfrak{G}) \cup T_r(\mathfrak{G}') \cup T_r(\mathfrak{G}''), \tag{14.16}$$

given in (6.8) of Ref. 5. The partition (14.16) implies the following open covering of $T_r(K)$ (Cf. (6.11), Ref. 5)

$$T_r(K) = T_r(K - \mathfrak{G}) \cup T_r(\mathfrak{H}') \cup T_r(\mathfrak{H}''). \tag{14.17}$$

In accord with (14.17), σ is overdefined on $T_r(K)$ by setting

(14.18)′	$\sigma(\mathbf{x}) = \omega_e \cdot \alpha_r(\mathbf{x})$	$\mathbf{x} \in T_r(K - \mathfrak{G})$
(14.18)″	$\sigma(\mathbf{x}) = R^r \cdot T_r^{-1}(\mathbf{x})$	$\mathbf{x} \in T_r(\mathfrak{H}')$
(14.18)‴	$\sigma(\mathbf{x}) = R^{r-1} \cdot T_r^{-1}(\mathbf{x})$	$\mathbf{x} \in T_r(\mathfrak{H}'')$

In accord with (14.11) all points of $CG' - \mathfrak{P}$, not in the sets $T_r(K)$, are included in L, and hence in the open set

$$L^+ = L \cup \bigcup_{r=1}^{\infty} T_r(K - \bar{K}_d) \subset M. \tag{14.19}$$

Since $\alpha_r(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in T_r(K - \bar{K}_d)$, the above definition of σ, and in particular (14.18)′, is consistent with the partial representation of σ

$$\sigma(\mathbf{x}) = \omega_e(\mathbf{x}) \qquad (\mathbf{x} \in L^+). \tag{14.20}$$

Thus the partial representations (14.18) and (14.20) completely define σ on $CG' - \mathfrak{P}$.

It is to be noted that the domain for $\mathbf{x}$ in each of these partial representations is an open subset of E. As shown in Ref. 5, Lemma 7.2, σ so defined, is a homeomorphism of CG' into E, and defines a C^m-diffeomorphism of $CG' - \mathfrak{P}$ into E. Moreover, as stated in (7.3), Ref. 5,

$$\sigma \cdot T_r(K) = \mathring{J}\, \omega_e(\beta\, T_r(K)). \tag{14.21}$$

Definition of λ_ω. Since $\mathbf{a}$ is a C^∞-diffeomorphism of D onto H', a mapping λ_ω of H' into E can be given by defining $\lambda_\omega \cdot \mathbf{a}(\mathbf{z})$ for $\mathbf{z} \in D$.

To that end we refer to the partition, Cf. Ref. 5 (3.3),

$$D = \mathfrak{G}' \cup \sigma(D - G'), \tag{14.22}$$

and consistently overdefine λ_ω on H' by setting

$$\boxed{\begin{array}{lll} (14.23)' & \lambda_\omega \cdot \mathbf{a}(\mathbf{z}) = \omega \cdot \mathbf{a} \cdot \sigma^{-1}(\mathbf{z}) & (\mathbf{z} \in \sigma(D - G')) \\ (14.23)'' & \lambda_\omega \cdot \mathbf{a}(\mathbf{z}) = \mathbf{z} & (\mathbf{z} \in \mathfrak{L}') \end{array}}$$

See proof of Theorem 3.1 (ii) in Ref. 5.

The exceptional point $\mathbf{a}(\mathfrak{P})$ of λ_ω. Since $\sigma(\mathfrak{P}) = \mathfrak{P}$ and $\mathfrak{P} \in D - G'$, $\mathfrak{P} \in \sigma(D - G')$. Moreover, (14.23)$'$ implies that

$$\lambda_\omega \cdot \mathbf{a}(\mathfrak{P}) = \omega \cdot \mathbf{a}(\mathfrak{P}). \tag{14.24}$$

The point $\mathfrak{P}$ is the one point at which σ^{-1} may fail to be of class C^m when $m > 0$. Hence $\mathbf{a}(\mathfrak{P})$ is the one point at which λ_ω may fail to be of class C^m when $m > 0$.

As shown in Ref. 5, § 3, λ_ω as defined in (14.23) satisfies Theorem 13.1.

Résumé. Each mapping defined in this section has as domain of definition an open subset of E, except that ω_e is defined on the union of the open set M and $\mathfrak{P}$. Each mapping is a homeomorphism into E. Except for the definitions $\omega_e(\mathfrak{P}) = \sigma(\mathfrak{P}) = \mathfrak{P}$, the partial mappings defined in individual equations have also been over open sets. This has necessitated the use of overdefinition. In every case this has been self-consistent.

The choice of d in § 13 depends on ω. Once this constant is fixed the mappings $\mathbf{a}$, R^r, and T_{r+1} may be affirmed to be C^∞-diffeomorphisms with no other dependence on ω. They will be taken over without change in the next section.

The mappings ω_e, a_r, σ and finally λ_ω, as defined above, are determined by ω. The partial mappings in the individual numbered equations are all C^m-diffeomorphisms of their open domains of definition, except that the partial mapping (14.23)$'$ may not be of class C^m at $\mathfrak{P}$, nor λ_ω at $\mathbf{a}(\mathfrak{P})$.

§ 15. The t-transform of an elementary Γ_m-problem.

According to Corollary 12.1 an arbitrary Γ_m-problem is equivalent to an elementary Γ_m-problem

$$\mathbf{P} = (\Phi, L, \Gamma_m). \tag{15.1}$$

By definition of such a problem there are associated with $\mathbf{P}$ an $(n-1)$-sphere S_Q with center Q, such that

$$\Phi^p(\mathbf{x}) = \mathbf{x} \qquad (\mathbf{x} \in \mathring{J} S_Q, \ p \in \Gamma_m) \tag{15.2}$$

and a shell δ of the form s_ρ (Cf. (7.8)), $0 < \rho < 1$, with ρ independent of $p \in \Gamma_m$, such that for each $p \in \Gamma_m$, $\delta \subset L^p$. We can suppose that the radius r of S_Q is at most $1/4$ and that $J S_Q \subset \delta$.

In this section we shall define a **t**-transform of **P**, analogous to the t-transform of P as defined in § 13. A solution λ_Q of this transformed problem will then be defined, leading to a solution Λ_Φ in § 18 of **P**.

Methods. In defining λ_Q, accessory p-invariant mappings Ψ and Ω are required and defined. In this section λ_Q is defined but not adequately represented. The definition of λ_Q in terms of its p-sections is complete, but does not disclose in an a priori way the fact that λ_Q is a homeomorphism, or on what subdomain λ_Q is a C^m-diffeomorphism.

In § 16 we shall turn to the problem of representing λ_Q so as to disclose these essential properties. In showing that λ_Q is a homeomorphism, our basic aid is Lemma 1.1, knowing that the p-sections of λ_Q are homeomorphisms. In deriving the differentiability of λ_Q, the essential aid will be Lemma 1.2. In the problem of representing λ_Q in § 16, there is the related problem of defining and representing the subsets of $E \times \Gamma_m$ upon which λ_Q and its associated p-invariant mappings are defined. These subsets of $E \times \Gamma_m$ are defined in terms of their p-sections. In showing that these subsets of $E \times \Gamma_m$ are open, Lemma 2.4 is fundamental. When compositions of p-sections of p-invariant mappings enter, the Notational Lemma 2.5 is needed. When p-invariant homeomorphisms of open subsets of $E \times \Gamma_m$ enter, Lemma 2.3, generalizing the Brouwer theorem on the invariance of domain, is required.

We now proceed to introduce the elements which are needed to define λ_{Ω}. We shall write λ_{Ω}^{p} in place of $(\lambda_{\Omega})^{p}$.

The p-invariant mapping $\mathbf{t}$. Let t be the reflection of $E - Q$ in S_Q. Let $\mathbf{t}$ be the p-invariant C^{∞}-diffeomorphism of $(E - Q) \times \Gamma_m$ onto $(E - Q) \times \Gamma_m$ under which $(\mathbf{x}, p) \rightarrow (t(\mathbf{x}), p)$.

The p-invariant mapping Ψ. Following the procedures of § 13, let Ψ be a p-invariant mapping of $t(\delta - Q) \times \Gamma_m$ into $E \times \Gamma_m$ such that

$$\Psi(\mathbf{t}(\mathbf{x}, p)) = \mathbf{t}(\Phi(\mathbf{x}, p)) \qquad (\mathbf{x} \in \delta - Q,\ p \in \Gamma_m). \tag{15.3}$$

Then Ψ is a C^m-diffeomorphism of $t(\delta - Q) \times \Gamma_m$ into $E \times \Gamma_m$. Equivalently Ψ can be defined by the conditions

$$\Psi^{p}(t(\mathbf{x})) = t(\Phi^{p}(\mathbf{x})) \qquad (\mathbf{x} \in \delta - Q,\ p \in \Gamma_m) \tag{15.4}$$

analogous to the condition on ψ in (13.2). We note that Ψ^p reduces to the identity on $E - J S_Q$ because of (15.2), and that the set $E - J S_Q$ is independent of $p \in \Gamma_m$.

The constant d. With the rectangles $K_{\nu d}$, $\nu = 1, 2, 3, 4$, defined as in § 13, a constant $d > 0$ can be chosen so small that Ψ^p reduces to the identity on $(K - K_{4d})$ for each $p \in \Gamma_m$. The subset of $E \times \Gamma_m$ on which Ψ is undefined, and on which $x_n < 0$, has a p-section which is the closed subset ζ' of K given by (13.3), while the subset of $E \times \Gamma_m$ on which Ψ is undefined, and on which $x_n > 0$, has a p-section which is the closed subset ζ'' of K given by (13.4).

The rectangles $H' \supset L' \supset G' \supset \Theta'$ and $H'' \supset L'' \supset G'' \supset \Theta''$, are defined in terms of K and d as in § 13. As previously, d will be chosen so that

$$\Theta' \supset \zeta' \quad \Theta'' \supset \zeta''. \tag{15.5}$$

The p-invariant mapping Ω. The analogue of ω in § 13 is the restriction ω^p of Ψ^p to $K - \Theta$. Let Ω be the p-invariant mapping of $(K - \Theta) \times \Gamma_m$ into $E \times \Gamma_m$ such that $\Omega^p = \omega^p$. One could also define Ω as the restriction of Ψ to $(K - \Theta) \times \Gamma_m$. It appears then that Ω is a C^m-diffeomorphism of $(K - \Theta) \times \Gamma_m$ into $E \times \Gamma_m$.

The rectangle D, point $\mathfrak{P}$, contraction $\mathbf{a}$ of D onto H', and the subrectangle D_0 of D are defined as in § 13.

Theorem 15.1 prepares for the basic Theorem 16.1.

Theorem 15.1. *There exists a p-invariant homeomorphism $\lambda_{\mathfrak{Q}}$ of $H' \times \Gamma_m$ into $E \times \Gamma_m$ which extends the restriction $\Omega | [(H' - \mathbf{a}(D_0)) \times \Gamma_m]$, which is such that $\lambda^p_{\mathfrak{Q}}(H') = \mathfrak{H}'_p$ for each $p \in \Gamma_m$ and which, when $m > 0$, is in addition a C^m-diffeomorphism of $(H' - \mathbf{a}(\mathfrak{P})) \times \Gamma_m$ into $E \times \Gamma_m$.*

The problem of finding a mapping $\lambda_{\mathfrak{Q}}$ which satisfies Theorem 15.1 will be called the **t**-*transform* of the elementary Γ_m-problem $\mathbf{P}$.

The given elementary Γ_m-problem (15.1) has a "p-section", by definition, of the form

$$\mathbf{P}^p = (\Phi^p, L^p, m) \qquad (p \in \Gamma_m). \tag{15.6}$$

One sees that $\mathbf{P}^p$ is a simple elementary problem. The t-transform of the elementary problem $\mathbf{P}^p$ is the simple problem

$$(\omega^p, H', m) \qquad (p \in \Gamma_m). \tag{15.7}$$

By virtue of Theorem 13.1 the simple problem (15.7) has a solution λ_{ω^p}, explicitly defined by formulas in § 14, for each $p \in \Gamma_m$, in terms of elements we now recall.

The mappings ω^p_e, α^p_r, σ^p. Corresponding to the mapping ω of $K - \Theta$ into E, mappings ω_e, α_r, σ into E have been formally defined in § 14 over the respective sets

$$M \cup \mathfrak{P}, \; T_r(K - \mathfrak{G}), \; CG' \qquad (r = 1, 2, \ldots).$$

When ω^p replaces ω it is necessary to replace the sets $\mathfrak{G}$, $\mathfrak{G}''$, $\mathfrak{G}'$, etc. by the sets,

$$\boxed{\begin{array}{ll} \mathfrak{G}'_p = J\omega^p(\beta G') & \mathfrak{G}''_p = J\omega^p(\beta G'') \\ \mathfrak{H}'_p = \mathring{J}\omega^p(\beta H') & \mathfrak{H}''_p = \mathring{J}\omega^p(\beta H'') \\ \mathfrak{L}'_p = \mathring{J}\omega^p(\beta L') & \mathfrak{L}''_p = \mathring{J}\omega^p(\beta L'') \\ \mathfrak{G}_p = \mathfrak{G}' \cup \mathfrak{G}''_p & \mathfrak{H}_p = \mathfrak{H}'_p \cup \mathfrak{H}''_p . \end{array}} \tag{15.8}$$

Mappings

$$\omega^p_e, \; \alpha^p_r, \; \sigma^p \qquad (p \in \Gamma_m) \tag{15.9}$$

naturally replace the mappings ω_e, α_r, σ when ω^p replaces ω, and are

similarly defined over the respective sets

$$M \cup \mathfrak{P}, \quad T_r(K-\mathfrak{G}_p), \quad CG' \qquad (r=1, 2, \ldots).$$

Definition. Suppose λ_{ω^p} defined as is λ_ω in (14.23), with $\omega^p, \sigma^p, \mathfrak{L}'_p$ replacing ω, σ, $\mathfrak{L}'$. Then λ_Ω will be defined as the p-invariant mapping of $H' \times \Gamma_m$ into $E \times \Gamma_m$ such that

$$\lambda_\Omega^p = \lambda_{\omega^p} \qquad (p \in \Gamma_m). \tag{15.10}$$

We state a major lemma.

Lemma 15.1. *The p-invariant mapping λ_Ω of $H' \times \Gamma_m$ into $E \times \Gamma_m$ defined by (15.10), satisfies Theorem 15.1.*

The mapping λ_Ω defined by (15.10) is biunique and obviously extends the restriction $\Omega \mid [(H'-\mathbf{a}(D_0)) \times \Gamma_m]$, since λ_{ω^p} extends the restriction $\omega^p \mid (H'-\mathbf{a}(D_0))$. Moreover $\lambda_\Omega^p(H') = \mathfrak{H}'_p$ in accord with Theorem 13.1 and the definition of $\mathfrak{H}'_p$ in (15.8). Cf. (14.13). To establish Lemma 15.1 it remains to prove the following lemma.

Lemma 15.2. *The mapping λ_Ω is a homeomorphism of $H' \times \Gamma_m$ into $E \times \Gamma_m$, and when $m > 0$, is in addition, a C^m-diffeomorphism into $E \times \Gamma_m$ of the set*

$$(H'-\mathbf{a}(\mathfrak{P})) \times \Gamma_m. \tag{15.11}$$

§ 16. The representation of λ_Ω.

The proof of Lemma 15.2 will be based on a representation of λ_Ω in terms of p-invariant mappings

$$\Omega_e, \ A_r, \ \Sigma \tag{16.1}$$

with p-sections

$$\omega_e^p, \ \alpha_r^p, \ \sigma^p \tag{16.2}$$

respectively. It is natural that the mappings (16.1) be used to represent λ_Ω since the mappings (16.2) were used to represent the p-section λ_{ω^p} of λ_Ω. In establishing the continuity and differentiability properties of Ω_e, A_r and Σ, affirmed in statements I, II, III below, the projections on E of these mappings will enter through our use of Lemmas 1.1 and 1.2. These projections will be denoted by Ω_{e1}, A_{r1}, Σ_1 respectively.

The p-invariant mapping Ω_e. The mapping ω_e^p is defined on the set $M \cup \mathfrak{P}$. We introduce the open subset of E

$$A^+ = A \cup [\bigcup_{r=0}^{\infty} R^r(K - \overline{K_d})] \qquad \text{(Cf. (14.0))} \tag{16.3}$$

in order that we may give M the infinite open covering

$$M = \bigcup_{r=0}^{\infty} R^r(K - \Theta) \cup A^+ \qquad \text{(Cf. (14.1))}. \tag{16.4}$$

Corresponding to this covering of M, ω_e^p admits the representation

$$\boxed{\begin{array}{lll} (16.5)' & \omega_e^p(\mathbf{x}) = R^r \cdot \omega^p \cdot R^{-r}(\mathbf{x}) & (\mathbf{x} \in R^r(K - \Theta)) \\ (16.5)'' & \omega_e^p(\mathbf{x}) = \mathbf{x} & (\mathbf{x} \in A^+) \end{array}}$$

on M in accord with the definition of ω_e^p in § 14.

The p-invariant mapping Ω_e is defined on $(M \cup \mathfrak{P}) \times \Gamma_m$ by putting $\Omega_e^p(\mathbf{x}) = \omega_e^p(\mathbf{x})$. We shall prove the following.

I. The p-invariant mapping Ω_e is a homeomorphism into $E \times \Gamma_m$ of $(M \cup \mathfrak{P}) \times \Gamma_m$ and, in addition, a C^m-diffeomorphism of $M \times \Gamma_m$ when $m > 0$.

Let $\mathbf{R}^r$, $r = 0, \pm 1, \pm 2, \ldots,$ be the p-invariant mapping of $E \times \Gamma_m$ onto $E \times \Gamma_m$ such that the point $(\mathbf{x}, p)$ in $E \times \Gamma_m$ goes into the point $(R^r(\mathbf{x}), p)$. Note that $\mathbf{R}^r$ is a C^m-diffeomorphism of $E \times \Gamma_m$ by Lemma 1.2, that $\mathbf{R}^0$ is the identity, and $\mathbf{R}^{-r}$ is the inverse of $\mathbf{R}^r$. In (16.5), $\omega_e^p(\mathbf{x}) = \Omega_{e1}(\mathbf{x}, p)$. On applying (2.20) of the Notational Lemma 2.5 to the right member of (16.5)′, one has

$$\boxed{\begin{array}{lll} (16.6)' & \Omega_{e1}(\mathbf{x}, p) = (\mathbf{R}^r \Omega \mathbf{R}^{-r})_1(\mathbf{x}, p) & ((\mathbf{x}, p) \in R^r(K - \Theta) \times \Gamma_m) \\ (16.6)'' & \Omega_{e1}(\mathbf{x}, p) = \mathbf{x} & ((\mathbf{x}, p) \in A^+ \times \Gamma_m) \end{array}}$$

on $M \times \Gamma_m$, provided the points $(\mathbf{x}, p)$ admitted in (16.6)′ are in the canonical domain B of $\mathbf{R}^r \Omega \mathbf{R}^{-r}$ defined in § 2. The points $(\mathbf{x}, p)$ admitted in (16.6)′ are in B in accord with Corollary 2.1 because the right member of (16.5)′ is well-defined. Since the domains in the partial representations (16.6) are open sets in $E \times \Gamma_m$, it follows that Ω_{e1} is of class C^m on $M \times \Gamma_m$. Thus Ω_e is a C^m-diffeomorphism on $M \times \Gamma_m$ (Lemma 1.2). To complete the proof of I it remains, in view of Lemma 1.1, to prove I (a).

I(a). The mapping Ω_{e1} is continuous at each point of $\mathfrak{P} \times \Gamma_m$.

Let $\mathbf{c}$ be the vector representing $\mathfrak{P}$ in E. In the proof of Lemma 5.1 of Ref. 5, a mapping $\mathbf{x} \to \mu(\mathbf{x})$ of M into R^+ is defined such that $\mu(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{c}$, and

$$\| \omega_e^p(\mathbf{x}) - \mathbf{c} \| \leq \mu(\mathbf{x}) \qquad (\mathbf{x} \in M,\ p \in \Gamma_m). \tag{16.7}$$

Since $\omega_e^p(\mathbf{c}) = \mathbf{c}$, and $\Omega_{e1}(\mathbf{x}, p) = \omega_e^p(\mathbf{x})$ by definition of Ω_e, the continuity of Ω_{e1} at each point of $\mathfrak{P} \times \Gamma_m$ follows from (16.7).

For future reference it is to be noted that the point $\omega_e^p(\mathbf{x})$ converges to $\mathbf{c}$ as $\mathbf{x} \to \mathbf{c}$, uniformly with respect to $p \in \Gamma_m$.

The p-invariant mapping A_r. The mapping α_r^p, $r = 1, 2, \dots$, is given (Cf. (14.14)) by

$$\alpha_r^p(\mathbf{x}) = T_r \cdot (\omega^p)^{-1} \cdot T_r^{-1}(\mathbf{x}) \qquad (\mathbf{x} \in T_r(K - \mathfrak{S}_p)). \tag{16.8}$$

Let Δ_r be the subset of $E \times \Gamma_m$ defined by its p-sections $(p \in \Gamma_m)$

$$\Delta_r^p = T_r(K) - T_r(\mathfrak{S}_p) \qquad (r = 1, 2, \dots). \tag{16.9}$$

The p-invariant mapping A_r is defined on its domain Δ_r by

$$A_r^p(\mathbf{x}) = \alpha_r^p(\mathbf{x}). \tag{16.10}$$

We need statement (i).

(i) The set Δ_r is open relative to $E \times \Gamma_m$ $(r = 1, 2, \dots)$.

Write $T_r(\mathfrak{S}_p)$ in the form (Cf. (15.8))

$$T_r(\mathfrak{S}_p) = J(T_r \omega^p)(\beta G') \cup J(T_r \omega^p)(\beta G''). \tag{16.11}$$

The subsets U and V of $E \times \Gamma_m$ whose p-sections are

$$U^p = CJ(T_r \omega^p)(\beta G'), \quad V^p = CJ(T_r \omega^p)(\beta G'')$$

are open in $E \times \Gamma_m$ in accord with Lemma 2.4, and (16.9) implies that

$$\Delta_r = (T_r(K) \times \Gamma_m) \cap U \cap V. \tag{16.12}$$

Statement (i) follows.

II. The p-invariant mapping A_r is a C^m-diffeomorphism of Δ_r into $E \times \Gamma_m$.

Let $\mathbf{T}_r$, $r = 1, 2, \dots$, be the p-invariant C^∞-diffeomorphism of $E \times \Gamma_m$ onto $E \times \Gamma_m$ such that $(\mathbf{x}, p)$ goes into $(T_r(\mathbf{x}), p)$. By (16.8), the

Notational Lemma 2.5, and Corollary 2.1, A_r can be given the representation

$$A_r(\mathbf{x}, p) = (\mathbf{T}_r \Omega^{-1} \mathbf{T}_r^{-1})(\mathbf{x}, p) \qquad ((\mathbf{x}, p) \in \Delta_r) \tag{16.13}$$

for $r = 1, 2, \ldots$. Statement II follows.

The p-invariant mapping Σ. The mapping σ^p is defined on the set CG'. The set $CG' - \mathfrak{P}$ admits the infinite open covering

$$CG' - \mathfrak{P} = \bigcup_{r=1}^{\infty} T_r(K) \cup L^+ \tag{16.14}$$

(Cf. (14.11) and (14.19)). Corresponding to this covering, and in accord with (14.18) and (14.20), σ^p admits the representation

(16.15)′	$\sigma^p(\mathbf{x}) = \omega_e^p \cdot \alpha_r^p(\mathbf{x})$	$(\mathbf{x} \in T_r(K - \mathfrak{S}_p))$
(16.15)″	$\sigma^p(\mathbf{x}) = R^r \cdot T_r^{-1}(\mathbf{x})$	$(\mathbf{x} \in T_r(\mathfrak{H}'_p))$
(16.15)‴	$\sigma^p(\mathbf{x}) = R^{r-1} \cdot T_r^{-1}(\mathbf{x})$	$(\mathbf{x} \in T_r(\mathfrak{H}''_p))$

on $T_r(K)$, and the representation

$$\sigma^p(\mathbf{x}) = \omega_e^p(\mathbf{x}) \qquad (\mathbf{x} \in L^+) \tag{16.16}$$

on L^+.

The p-invariant mapping Σ is defined on $CG' \times \Gamma_m$ by putting $\Sigma^p(\mathbf{x}) = \sigma^p(\mathbf{x})$. We shall represent Σ_1 on the set $(CG' - \mathfrak{P}) \times \Gamma_m$ in a form which corresponds to the representations of the p-sections of Σ in (16.15) and (16.16).

Let Δ'_r, Δ''_r, $r = 1, 2, \ldots$, be the subsets of $E \times \Gamma_m$ defined by their p-sections $(p \in \Gamma_m)$

$$(\Delta'_r)^p = T_r(\mathfrak{H}'_p), \qquad (\Delta''_r)^p = T_r(\mathfrak{H}''_p). \tag{16.17}$$

It follows from Lemma 2.4 that Δ'_r, Δ''_r are open relative to $E \times \Gamma_m$, and from (14.17) that

$$\Delta_r \cup \Delta'_r \cup \Delta''_r = T_r(K) \times \Gamma_m. \tag{16.18}$$

By (16.15), (16.16), the Notational Lemma 2.5 and Corollary, Σ_1 can be given the representation

(16.19)′	$\Sigma_1(\mathbf{x}, p) = (\Omega_e A_r)_1(\mathbf{x}, p)$	$((\mathbf{x}, p) \in \Delta_r)$
(16.19)″	$\Sigma_1(\mathbf{x}, p) = (\mathbf{R}^r \mathbf{T}_r^{-1})_1(\mathbf{x}, p)$	$((\mathbf{x}, p) \in \Delta'_r)$
(16.19)‴	$\Sigma_1(\mathbf{x}, p) = (\mathbf{R}^{r-1} \mathbf{T}_r^{-1})_1(\mathbf{x}, p)$	$((\mathbf{x}, p) \in \Delta''_r)$

on $T_r(K) \times \Gamma_m$, and the representation

$$\Sigma_1(\mathbf{x}, p) = \Omega_{e1}(\mathbf{x}, p) \qquad ((\mathbf{x}, p) \in L^+ \times \Gamma_m) \tag{16.20}$$

on $L^+ \times \Gamma_m$.

III. The p-invariant mapping Σ is a homeomorphism into $E \times \Gamma_m$ of $CG' \times \Gamma_m$ and, in addition, a C^m-diffeomorphism of $(CG' - \mathfrak{P}) \times \Gamma_m$ when $m > 0$.

In accord with (16.14) and (16.18), the partial representations (16.19), (16.20) completely define Σ_1 on the set $(CG' - \mathfrak{P}) \times \Gamma_m$. Since the right members in these representations are of class C^m on their respective domains and these domains are open subsets of $E \times \Gamma_m$, Σ_1 is of class C^m on $(CG' - \mathfrak{P}) \times \Gamma_m$. Thus Σ is a C^m-diffeomorphism of $(CG' - \mathfrak{P}) \times \Gamma_m$ (Lemma 1.2). To complete the proof of III it remains, in view of Lemma 1.1, to prove III (a).

III (a). The mapping Σ_1 is continuous at each point of $\mathfrak{P} \times \Gamma_m$. The proof of III (a) is similar to that of I (a). Paralleling (16.7) we affirm that there exists a mapping $\mathbf{x} \to \nu(\mathbf{x})$ of a neighborhood N of $\mathfrak{P}$ relative to E, into the interval $[0, \infty)$, such that $\nu(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{c}$ and

$$\| \sigma^p(\mathbf{x}) - \mathbf{c} \| \leqq \nu(\mathbf{x}) \qquad (\mathbf{x} \in N,\ p \in \Gamma_m). \tag{16.21}$$

One can set $\nu(\mathbf{x}) = \mu(\mathbf{x})$ at all points in N at which $\sigma^p(\mathbf{x}) = \omega^p_e(\mathbf{x})$. Set $\nu(\mathbf{c}) = 0$. If N is sufficiently small the remaining points in N are in sets $T_r(K)$. In accord with (14.21)

$$\sigma^p T_r(K) = \mathring{J}\, \omega^p_e(\beta T_r(K)). \tag{16.22}$$

Since $\omega^p_e(\mathbf{x})$ converges to $\mathbf{c}$, as $\mathbf{x} \to \mathbf{c}$, uniformly with respect to $p \in \Gamma_m$, (16.22) implies that $\sigma^p T_r(K)$ converges to $\mathbf{c}$ as $r \uparrow \infty$, uniformly with respect to $p \in \Gamma_m$. The definition of $\nu(\mathbf{x})$ on N can accordingly be completed so that (16.21) holds and $\nu(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{c}$.

Statement III (a) follows and the proof of III is complete.

Proof of Lemmas 15.1 and 15.2. The relations (14.23), with ω^p, σ^p, $\mathfrak{L}'_p$ replacing ω, σ, $\mathfrak{L}'$ give the following for each $p \in \Gamma_m$.

$$\boxed{\begin{array}{lll} (16.23)' & \lambda_{\omega^p} \cdot \mathbf{a}(\mathbf{z}) = \omega^p \cdot \mathbf{a} \cdot (\sigma^p)^{-1}(\mathbf{z}) & (\mathbf{z} \in \sigma^p(D - G')) \\ (16.23)'' & \lambda_{\omega^p} \cdot \mathbf{a}(\mathbf{z}) = \mathbf{z} & (\mathbf{z} \in \mathfrak{L}'_p). \end{array}}$$

To establish Lemma 15.2 we need an adequate representation of the E-projection $\lambda_{\mathfrak{Q}1}$ of $\lambda_{\mathfrak{Q}}$. This is obtained from (16.23) as follows.

The subsets W' and W'' of $E \times \Gamma_m$. Let W' and W'' be subsets of $E \times \Gamma_m$ such that

$$(W')^p = \sigma^p (D - G'), \quad (W'')^p = \mathfrak{Q}'_p \qquad (p \in \Gamma_m).$$

The set $W' = \Sigma[(D - G') \times \Gamma_m]$, and so is open by Lemma 2.3. The set W'' is open as a consequence of Lemma 2.4.

The p-invariant mapping $\mathbf{A}$. Let $\mathbf{A}$ be the p-invariant C^∞-diffeomorphism of $D \times \Gamma_m$ onto $H' \times \Gamma_m$ under which $(\mathbf{x}, p)$ goes into $(\mathbf{a}(\mathbf{x}), p)$.

It follows from the definition of $\lambda_{\mathfrak{Q}}$ in (15.10), and from (16.23), using the Notational Lemma 2.5 and Corollary, that

$$\boxed{\begin{aligned} &[\lambda_{\mathfrak{Q}} \mathbf{A}]_1 (\mathbf{z}, p) = [\Omega \mathbf{A} \Sigma^{-1}]_1 (\mathbf{z}, p) \quad &&((\mathbf{z}, p) \in W') \qquad &&(16.24)' \\ &[\lambda_{\mathfrak{Q}} \mathbf{A}]_1 (\mathbf{z}, p) = \mathbf{z} &&((\mathbf{z}, p) \in W''). &&(16.24)'' \end{aligned}}$$

Now $\Omega \mathbf{A} \Sigma^{-1}$ is a p-invariant homeomorphism of W' into $E \times \Gamma_m$, which, when $m > 0$, defines, in addition, a C^m-diffeomorphism of the open subset $W' - (\mathfrak{P} \times \Gamma_m)$ of W' into $E \times \Gamma_m$. Moreover W' and W'' are open subsets of $E \times \Gamma_m$ whose union is $D \times \Gamma_m$ (cf. (14.22)). With the aid of Lemmas 1.1 and 1.2, (16.24) implies that $\lambda_{\mathfrak{Q}} \mathbf{A}$ is a p-invariant homeomorphism of $D \times \Gamma_m$ into $E \times \Gamma_m$, which, when $m > 0$, defines, in addition, a C^m-diffeomorphism of $(D - \mathfrak{P}) \times \Gamma_m$ into $E \times \Gamma_m$. Taking account of the nature of $\mathbf{A}$ and its inverse, we see that $\lambda_{\mathfrak{Q}}$ satisfies Lemma 15.2 and hence Lemma 15.1.

This completes the proof of Theorem 15.1.

Exceptional loci. Let A be an open subset of a differentiable manifold Σ of class C^m, $m > 0$. Let B be a closed subset of A, and η a homeomorphism of A into Σ which is a C^m-diffeomorphism into Σ of $A - B$. We say then that B is an m-exceptional set of η. If B is an m-exceptional set of η, then B' is also an m-exceptional set of η, provided B' is closed and $B' \supset B$. Our definition does not imply that an m-exceptional set is minimal. The subset B of A might be affirmed to be an m-exceptional set for each mapping of a class $[\eta]$ of mappings η which includes a C^m-diffeomorphism η_1 of A into Σ.

For the mapping Λ_Φ of Theorem 1.2, $Z\times\Gamma_m$ is an m-exceptional set. For the mapping $\lambda_{\mathfrak{Q}}$ of Theorem 15.1, $\mathbf{a}(\mathfrak{P})\times\Gamma_m$ is an m-exceptional set. In Lemma 17.1 $w\times\Gamma_m$ is an m-exceptional set of F_Φ, where w may differ from Z. In proving Lemma 17.1 we shall make use of the following lemma.

Lemma 16.1. Given the mapping $\eta: A\rightarrow\Sigma$ as above, let $B\subset A$ be an m-exceptional set of η. (i) If f is a C^m-diffeomorphism into Σ of an open subset of Σ such that the range of f includes B, then $f^{-1}(B)$ is an m-exceptional set of ηf. (ii) If F is a C^m-diffeomorphism into Σ of an open subset of Σ which includes $\eta(B)$, then B is an m-exceptional set of $F\eta$.

§17. Lemma 17.1, prelude to Theorem 1.2.

Lemma 17.1 will be derived from Theorem 15.1, and, as we shall show in §18, will imply Theorem 1.2. Lemma 17.1 differs from Theorem 1.2 in that Lemma 17.1 implies that $w\times\Gamma_m$ is an m-exceptional set of F_Φ, whereas Theorem 1.2 implies that $Z\times\Gamma_m$ is an m-exceptional set of Λ_Φ. Moreover Lemma 17.1 deals with an elementary Γ_m-problem rather than with the general Γ_m-problem with which Theorem 1.2 is concerned.

Lemma 17.1. Let $\mathbf{P}=(\Phi, L, \Gamma_m)$ be the arbitrary elementary Γ_m-problem (15.1). If $\varkappa$ is a suitably chosen compact subset of $\mathring{J}S$, which, when $m>0$, includes a suitably chosen point w, there exists a p-invariant homeomorphism F_Φ of $L\cup(JS\times\Gamma_m)$ into $E\times\Gamma_m$, which extends $\Phi|\xi$, where $\xi=L-(\varkappa\times\Gamma_m)$, and when $m>0$, is, in addition, a C^m-diffeomorphism into $E\times\Gamma_m$ of

$$L\cup(JS\times\Gamma_m)-(w\times\Gamma_m). \tag{17.0}$$

Definition of M', M'', M'''. Recall the set inclusions (cf. §13),

$$(\mathbf{x}\mid x_n<0)\supset H'\supset\mathbf{a}(D_0)\supset\zeta'. \tag{17.1}$$

Recall that $\mathring{J}S=t(\mathbf{x}\mid x_n<0)$, and set

$$M'=t(\beta H'),\quad M''=t(\beta\mathbf{a}(D_0)),\quad M'''=t(\beta\zeta'). \tag{17.2}$$

Since t maps JM', JM'' and JM''' homeomorphically into E, (17.2)

implies that

$$\mathring{J}M' = t(H'), \quad JM'' = t(\mathbf{a}(D_0)), \quad JM''' = t(\zeta'). \tag{17.3}$$

On applying t to the successive sets in (17.1) one finds that

$$\mathring{J}S \supset \mathring{J}M' \supset JM'' \supset JM'''. \tag{17.4}$$

If δ is the spherical shell introduced in § 15, $t(\zeta') = \mathring{J}S - \delta$, by virtue of the definition of ζ' in (13.3). Hence $JM''' = \mathring{J}S - \delta$ in accord with the last equality in (17.3). It follows then from (17.4) that

$$\mathring{J}S - \delta = \mathring{J}M' - \delta = JM'' - \delta = JM'''. \tag{17.5}$$

The domain $L^p \cup \mathring{J}S$ of F^p_Φ. The shell δ was chosen in § 15 so that $L^p \supset \delta$ for each $p \in \Gamma_m$. On taking the union of L^p with the successive sets in (17.5), and using the inclusion $L^p \supset \delta$, one obtains the relations

$$L^p \cup \mathring{J}S = L^p \cup \mathring{J}M' = L^p \cup JM'' = L^p \cup JM'''. \tag{17.6}$$

The restriction $F^p_\Phi \mid (\mathring{J}M' - JM'')$, $p \in \Gamma_m$. Before coming to the definition of F_Φ we shall deduce a relation that links the extension problem on $H' \times \Gamma_m$, as solved in Theorem 15.1, with the problem **P**. Since $\delta - Q \supset \mathring{J}M' - JM''$, the relation (15.3) which defines Ψ over $t(\delta - Q) \times \Gamma_m$, holds in particular for $\mathbf{x} \in \mathring{J}M' - JM''$ and $p \in \Gamma_m$. In accord with (17.3) $t(\mathring{J}M' - JM'') = H' - \mathbf{a}(D_0)$ so that Theorem 15.1 implies that

$$\omega^p(\mathbf{y}) = \lambda^p_\Omega(\mathbf{y}), \qquad (\mathbf{y} \in t(\mathring{J}M' - JM'')). \tag{17.7)'$$

It follows from the definition of ω^p and (17.7) that

$$\Phi^p(\mathbf{x}) = t \cdot \omega^p \cdot t(\mathbf{x}) = t \cdot \lambda^p_\Omega \cdot t(\mathbf{x}) \qquad (\mathbf{x} \in \mathring{J}M' - JM''). \tag{17.7)''$$

Lemma 17.2. *The p-invariant mapping*

$$F_\Phi : L \cup (\mathring{J}S \times \Gamma_m) \to E \times \Gamma_m \tag{17.8}$$

defined by the conditions

$$\boxed{\begin{array}{lll} F^p_\Phi(\mathbf{x}) = \Phi^p(\mathbf{x}) & (\mathbf{x} \in L^p - JM'') & (17.9)' \\ F^p_\Phi(\mathbf{x}) = t \cdot \lambda^p_\Omega \cdot t(\mathbf{x}) & (\mathbf{x} \in \mathring{J}M') & (17.9)'' \end{array}}$$

for each $p \in \Gamma_m$, taken with a set $\varkappa$ and point w such that

$$\varkappa = t(\mathbf{a}(D_0)) = J(M''), \quad w = t(\mathbf{a}(\mathfrak{P})),$$

satisfies Lemma 17.1.

We begin the proof of Lemma 17.1 by proving statements (a), (b), (c) for each $p \in \Gamma_m$.

(a) The composition of mappings in the right member of (17.9)″ is well-defined.

(b) The union of the domains in (17.9)′ and (17.9)″ is the set $L^p \cup \mathring{J}S$. On this set F^p_Φ is overdefined, but consistently.

(c) The mapping F^p_Φ of $L^p \cup \mathring{J}S$ into E is biunique.

Proof of (a). Since $\mathring{J}M' \subset \mathring{J}S$, $\mathring{J}M'$ does not contain the singular point Q of t. Hence $t(\mathbf{x})$ is well-defined in (17.9)″. We shall now verify the fact that when $\mathbf{x} \in \mathring{J}M'$ and $\mathbf{y} = t(\mathbf{x})$ then $\lambda^p_\Omega(\mathbf{y})$ is defined, and that, when $\mathbf{z} = \lambda^p_\Omega \cdot t(\mathbf{x})$, $t(\mathbf{z})$ is defined. Now $t(\mathring{J}M') = H'$ in accord with (17.3) and $\lambda^p_\Omega(H') = \mathfrak{H}'_p$ by virtue of Theorem 15.1. The mapping t is defined over $\mathfrak{H}'_p$ if $\mathfrak{H}'_p$ does not meet Q. That $\mathfrak{H}'_p$ does not meet Q follows from statement (π) of § 7, Ref. 5.

Proof of (b). It is a consequence of (17.4) and (17.6) that

$$(L^p - JM'') \cup \mathring{J}M' = L^p \cup \mathring{J}M' = L^p \cup \mathring{J}S \qquad (p \in \Gamma_m). \tag{17.10}$$

Moreover the equations (17.9) overdefine F^p_Φ on the set

$$(L^p - JM'') \cap \mathring{J}M' = (\mathring{J}M' - JM'') \cap L^p = \mathring{J}M' - JM''. \tag{17.11}$$

It follows from (17.7)″ that the representations (17.9)′ and (17.9)″ are consistent for $\mathbf{x} \in \mathring{J}M' - JM''$.

Proof of (c). We begin the proof of (c) by showing that (i) the restriction of F^p_Φ to $\mathring{J}S$ is biunique for each $p \in \Gamma_m$.

To this end we use Corollary 4.1 of Ref. 5, satisfying the conditions of this corollary by setting $f = \Phi^p$, $\Delta = L^p$ and

$$X = JM', \quad Y = \mathring{J}S, \quad \mathfrak{X}_p = J\Phi^p(M'), \quad \mathfrak{Y}_p = \mathring{J}\Phi^p(S), \tag{17.12}$$

where $\mathfrak{X}_p$ and $\mathfrak{Y}_p$ here replace $\mathfrak{X}$ and $\mathfrak{Y}$ in the corollary. Conditions (α) and (β) of the corollary are trivially satisfied. Condition (γ) is satisfied, since Φ^p maps points of L^p which are interior to S into points which are interior to $\Phi^p(S)$. According to the corollary, one has the partition

$$\mathfrak{Y}_p = \mathring{\mathfrak{X}}_p \cup \Phi^p(Y - \mathring{X}). \tag{17.13}$$

To establish (c), the mapping F^p_Φ will be applied to the sets $\mathring{X}$ and $Y - \mathring{X}$ in the partition $Y = \mathring{X} \cup (Y - \mathring{X})$. Under F^p_Φ, $\mathring{X}$ is mapped biuniquely onto $\mathring{\mathfrak{X}}_p$; for

$$\begin{aligned} F^p_\Phi(\mathring{X}) &= t \cdot \lambda^p_\Omega(\mathring{X}) = t \cdot \lambda^p_\Omega(H') = t\mathfrak{H}'_p \\ &= t(\mathring{J}\omega^p(\beta H')) = \mathring{J}(t \cdot \omega^p \cdot t(M')) = \mathring{J}\Phi^p(M') = \mathring{\mathfrak{X}}_p. \end{aligned}$$

On the other hand F^p_Φ maps $Y - \mathring{X}$ biuniquely onto $\Phi^p(Y - \mathring{X})$, since $(17.9)'$ holds and Φ^p is biunique. Since the two sets on the right of (17.13) do not meet we infer that $F^p_\Phi | Y$ is biunique.

To complete the proof of (c) note first that $F^p_\Phi(Y) = \mathfrak{Y}_p = \mathring{J}\Phi^p(S)$ in accord with (17.13) and the definition of $\mathfrak{Y}_p$. We now give the domain $L^p \cup \mathring{J}S$ of F^p_Φ the partition $(L^p - Y) \cup Y$. Recall that Φ^p, and hence F^p_Φ, maps $L^p - Y$ biuniquely (by hypothesis) into the closed exterior of $\Phi^p(S)$, and hence onto a set disjoint from the set $\mathring{J}\Phi^p(S)$ onto which F^p_Φ maps Y.

This establishes (c).

The representation (17.14) of F_Φ. In order to complete the proof of Lemma 17.1 we shall give a representation of F_Φ which reveals the continuity and differentiability of F_Φ.

Recall the p-invariant C^∞-diffeomorphism $\mathbf{t}$ introduced in § 15. It follows from the Notational Lemma 2.5 and Corollary that F_Φ, as defined in (17.9), can be given the representation

$$\boxed{\begin{array}{lll} F_\Phi(\mathbf{x}, p) = \Phi(\mathbf{x}, p) & ((\mathbf{x}, p) \in \xi) & (17.14)' \\ F_\Phi(\mathbf{x}, p) = (\mathbf{t}\lambda_\Omega \mathbf{t})(\mathbf{x}, p) & ((\mathbf{x}, p) \in \mathring{J}M' \times \Gamma_m), & (17.14)'' \end{array}}$$

where $\xi = L - (JM'' \times \Gamma_m) = L - (\varkappa \times \Gamma_m)$.

Proof of Lemma 17.1. The case $m = 0$. We note that the union of the domains in (17.14) is

$$L \cup (\mathring{J}M' \times \Gamma_m) = L \cup (\mathring{J}S \times \Gamma_m) \qquad (\text{Cf. } (17.6)). \tag{17.15}$$

The p-section of the set (17.15) is the domain of definition $L^p \cup \mathring{J}S$ of F^p_Φ in Lemma 17.2. That F_Φ is biunique follows from the biuniqueness of

each of its sections F_{Φ}^{p} as defined in (17.9). The domains in (17.14)′ and (17.14)″ give an open covering of the domain (17.15) of F_{Φ}. It follows from (17.14) that $F_{\Phi 1}$ is continuous over the domain (17.15), so that by Lemma 1.1, F_{Φ} is a homeomorphic mapping of the set (17.15) into $E \times \Gamma_m$. Finally F_{Φ} extends $\Phi | \xi$, by virtue of (17.14)′.

The case $m>0$. It remains to show that $w \times \Gamma_m$ is an m-exceptional set of F_{Φ} in case $m>0$.

To that end we shall apply Lemma 16.1. The point $\mathbf{a}(\mathfrak{P})$ is in H', so that $t\mathbf{a}(\mathfrak{P})$ is in the set $tH' = \mathring{J}M'$. Cf. (17.3). The image under F_{Φ} of a pair (w, p) for $p \in \Gamma_m$ is accordingly given by (17.14)″. Now $\mathbf{a}(\mathfrak{P}) \times \Gamma_m$ is an m-exceptional set of $\lambda_{\mathfrak{Q}}$ in accord with Theorem 15.1. By Lemma 16.1 (i), the set

$$\mathbf{t}(\mathbf{a}(\mathfrak{P}) \times \Gamma_m) = t(\mathbf{a}(\mathfrak{P})) \times \Gamma_m = w \times \Gamma_m$$

is an m-exceptional set of $\lambda_{\mathfrak{Q}}\mathbf{t}$, restricted as in (17.14)″. On this same domain $\mathbf{t}\lambda_{\mathfrak{Q}}\mathbf{t}$ is well-defined and has $w \times \Gamma_m$ as an m-exceptional set, by (ii) of Lemma 16.1. It follows, using (17.14) and Lemma 1.2, that F_{Φ} is a C^m-diffeomorphism into $E \times \Gamma_m$ of the set (17.0).

This completes the proof of Lemma 17.1.

§ 18. Proof of Theorem 1.2.

We begin with a proof of the following.

(α) Theorem 1.2 holds for the case in which the problem (Φ, L, Γ_m) is elementary.

Referring to Lemma 17.1 we shall make use of the set $\varkappa$, point w and mapping F_{Φ}. Recall also the shell δ which was so chosen in § 15 that $\delta \subset L^p$ for each $p \in \Gamma_m$. Let N_* be an open neighborhood of S such that (i) the line segment joining w to the center Z of S does not meet ClN_*, and (ii) $N_* \subset \delta - \varkappa$. It follows from (i) and Lemma 2.1 of Ref. 14 that there exists a C^∞-diffeomorphism τ of E onto E that leaves $N_* \cup CJS$ pointwise invariant and carries Z into w. Let ζ be the p-invariant C^∞-diffeomorphism of $E \times \Gamma_m$ onto $E \times \Gamma_m$ such that for arbitrary $\mathbf{x} \in E$ and $p \in \Gamma_m$, $\zeta(\mathbf{x}, p) = (\tau(\mathbf{x}), p)$. The domain $L \cup (JS \times \Gamma_m)$ of F_{Φ} remains

invariant as a set under the mapping ζ. We shall show that the p-invariant homeomorphism

$$\Lambda_\Phi = F_\Phi G \qquad (G = \zeta \mid L \cup (JS \times \Gamma_m)) \tag{18.1}$$

of $L \cup (JS \times \Gamma_m)$ into $E \times \Gamma_m$, taken with the neighborhood $L_* = N_* \times \Gamma_m$ of $S \times \Gamma_m$ relative to $E \times \Gamma_m$, satisfies Theorem 1.2.

Recalling that $L_e = L - (JS \times \Gamma_m)$ by definition, we shall first show that

$$\Lambda_\Phi \mid (L_* \cup L_e) = \Phi \mid (L_* \cup L_e). \tag{18.2}$$

To that end observe that the set $(N_* \cup CJS) \times \Gamma_m$ on which ζ reduces to the identity, includes $L_* \cup L_e$. Hence Λ_Φ reduces to F_Φ on $L_* \cup L_e$. Since F_Φ reduces to Φ on $L - (\varkappa \times \Gamma_m)$ the inclusion

$$L_* \cup L_e \subset L - (\varkappa \times \Gamma_m) \tag{18.3}$$

implies that F_Φ reduces to Φ on $L_* \cup L_e$. Relation (18.2) follows.

We shall now show that $Z \times \Gamma_m$ is an m-exceptional set of Λ_Φ. The mapping F_Φ is a p-invariant homeomorphism into $E \times \Gamma_m$ of $L \cup (JS \times \Gamma_m)$. Moreover $w \times \Gamma_m$ is an m-exceptional set of F_Φ in accord with Lemma 17.1. Now $\zeta(Z \times \Gamma_m) = w \times \Gamma_m$, and it then follows from Lemma 16.1 that $Z \times \Gamma_m$ is an m-exceptional set of Λ_Φ.

This completes the proof of (α).

An arbitrary Γ_m-problem $\mathbf{P}^*$ has the form

$$\mathbf{P}^* = f' \mathbf{P} f \qquad \text{(Cor. 12.1)} \tag{18.4}$$

where $\mathbf{P} = (\Phi, L, \Gamma_m)$ is an elementary Γ_m-problem, where f' operates externally on $\mathbf{P}$ and f is a reduced mapping of $E \times \Gamma_m$ onto $E \times \Gamma_m$. If (Λ_Φ, L_*) affords a solution of problem $\mathbf{P}$ then by Theorem 4.1 $(f' \Lambda_\Phi, L_*)$ is a solution of problem $f' \mathbf{P}$, and, by Theorem 5.1,

$$(f' \Lambda_\Phi f, f^{-1}(L_*)) \tag{18.5}$$

is a solution of the given Γ_m-problem $\mathbf{P}^*$.

This completes the proof of Theorem 1.2.

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SUR UN THÉORÈME DE PROLONGEMENT FONCTIONNEL DE KELDYCH CONCERNANT LE PROBLÈME DE DIRICHLET [1]

Par

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à Paris, France

I. INTRODUCTION

1. Considérons un ouvert borné ω de R^n et l'ensemble $\mathscr{C}$ des fonctions réelles finies continues sur la frontière ω^*. On dit que $f \in \mathscr{C}$ est classiquement résolutive s'il existe une fonction harmonique dans ω tendant en tout point-frontière x_0 vers $f(x_0)$. On notera $\mathscr{C}_R$ l'ensemble de telles f. On connaît d'autre part pour toute $f \in \mathscr{C}$ la solution généralisée $H_f(x)$ de Perron-Wiener qui vaut la solution classique lorsque $f \in \mathscr{C}_R$. Or Keldych a montré ([15]—[16]) que si $\mathscr{L}_f(x)$ est harmonique par rapport à $x \in \omega$, et pour tout $x \in \omega$, fonctionnelle croissante [2] de f, égale à $H_f(x)$ pour $f \in \mathscr{C}_R$, alors $\mathscr{L}_f(x) = H_f(x)$ pour toute $f \in \mathscr{C}$. C'est une conséquence facile d'un lemme affirmant l'existence pour tout point-frontière régulier $x_0 \in \omega^*$ d'une fonction $\varphi \in \mathscr{C}_R$ telle que

$$\varphi(x_0) = 0, \quad \varphi(x) > 0 \quad \forall x \in \omega^* - \{x_0\}.$$

Mais ce lemme était établi par Keldych d'une manière compliquée, basée sur le critère de régularité de Wiener et qui ne peut s'étendre aux problèmes de Dirichlet très généraux considérés aujourd'hui.

Je voudrais d'abord faire remarquer qu'il suffit d'un lemme plus faible, facile à démontrer (et dont on peut d'ailleurs déduire le lemme de Keldych). Puis, j'examinerai l'extension de ces développements pour le problème de Dirichlet dans une axiomatique très générale que j'ai récemment développée ([4], [5], [6]) et dont Mme Hervé poursuit l'étude ([10], [11], [12], [13]). Il faudra une certaine adaptation des démonstrations et une petite restriction indispensable sur $\mathscr{L}_f$ ou sur la frontière.

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1. Ces recherches ont fait l'objet d'exposés, d'abord au Séminaire de théorie du potentiel (Paris, janvier 1960).

2. En fait, Keldych impose $\mathscr{L}_f \leq \sup f$, ce qui, avec la linéarité, entraîne la croissance. Mais la croissance est seule utilisée, par suite seule supposée ici.

Enfin, au lieu de s'occuper de la croissance de $\mathscr{L}_f$, on peut songer à imposer la linéarité ou même seulement la continuité pour une topologie convenable sur $\mathscr{C}$. Je donnerai donc sur le prolongement fonctionnel de la solution classique H_f $(f \in \mathscr{C}_R)$ selon H_f $(f \in \mathscr{C})$ des résultats de ce genre, basés sur une démonstration analogue à celle du nouveau lemme et valables dans le cadre de l'axiomatique générale considérée.

II. LE LEMME AFFAIBLI

2. Lemme 1. *Soit x_0 un point-frontière régulier d'un ouvert borné ω de R^n, δ un voisinage de x_0, ε et K choisis >0. Il existe sur ω^* une fonction $\varphi \in \mathscr{C}_R$ satisfaisant à $\varphi \geqq 0$, $\varphi(x_0) < \varepsilon$, et $\varphi(x) \geqq K$ pour $x \in \omega^* \cap \complement\delta$.*

On peut faire en sorte que $\varphi \leqq K'$ (K' choisi $> K$)[3] et on la réalise en définissant une fonction sous-harmonique finie continue $u \geqq 0$ dans $\omega_0 \supset \overline{\omega}$, harmonique dans ω, satisfaisant à $u(x_0) < \varepsilon$, $u(x) \geqq K$ pour $x \in \complement\delta$, et $u \leqq K'$ dans ω_0.

Prenons pour ω_0 une boule de centre x_0 et introduisons une autre boule b_r de centre x_0 et rayon r; la solution du problème de Dirichlet dans la couronne $\omega_0 - \overline{b_r}$ avec donnée K' sur ω_0^* et 0 sur b_r^* tend vers K' quand $r \to 0$, donc majore K dans $\complement\delta$ pour r convenable. Prolongée par 0 dans $\overline{b_r}$, cette solution donne une fonction sous-harmonique U $(0 \leqq U < K')$. Le "balayage" de ω qui consiste à remplacer dans ω, U par H_U^ω, puis la nouvelle fonction dans ω_0 par sa limite supérieure en chaque point (ce qui revient à une majoration aux points-frontière irréguliers de ω), donne une fonction sous-harmonique $u \geqq U$, harmonique dans ω et telle que $u \geqq 0$, $u < K'$ et $u(x_0) = 0$, $u(x) \geqq K$ hors δ.

Noter que $v = K' - u$ est le potentiel de Green d'une mesure $\mu \geqq 0$ dans ω_0 (portée par $\omega^* \cup b_r^*$). Or, d'après un théorème classique de Lusin, il existe un ouvert σ de μ-mesure $< \alpha$ (donné >0) tel que la restriction de v sur $\complement\sigma$ soit continue; le potentiel V des masses situées hors σ a donc sa restriction sur $\complement\sigma$ aussi continue[4]; par suite V est continu dans

3. Je dois à Choquet une remarque ayant permis de satisfaire à cette condition, indispensable pour retrouver le lemme de Keldych lui-même.

4. On rappelle que si la somme de deux fonctions semi-continues inférieurement et $>-\infty$ est continue en un point, chacune des fonctions est continue en ce point.

R^n [5]. De plus, le potentiel $v - V$ des masses situées sur σ est en x_0 arbitrairement petit avec α.

On obtiendrait aussi un tel V en remarquant que les points-frontière irréguliers de ω forment un ensemble de μ-mesure nulle et en prenant pour σ un ouvert de μ-mesure $< \alpha$, contenant ces points irréguliers mais non x_0. On voit directement que v est continu hors σ, donc le potentiel des masses situées hors σ est continu partout.

Du potentiel V obtenu, on déduit $K' - V$ qui répond à la question.

3. Application au lemme de Keldych.

On peut déduire de là le lemme de Keldych comme me l'a communiqué H. Bauer. Voici une démonstration analogue :

On va former une fonction surharmonique > 0 dans la boule ω_0, finie continue, égale à 1 en x_0 régulier de ω^*, < 1 ailleurs et harmonique dans ω.

On part de v_1 surharmonique continue dans ω_0 et satisfaisant à $0 < v_1 < 1$, harmonique dans ω.

Soit δ une boule de centre x_0 où $v_1 < v_1(x_0) + \frac{1}{2}(1 - v_1(x_0))$. On peut former une fonction w de même type que v_1, satisfaisant à :

$$\begin{cases} 0 < w < \frac{1}{2}(1 - v_1(x_0)) & \text{partout} \\ w < \alpha(1 - \sup\limits_{C\delta} v_1) \quad \text{dans} \quad C\delta & (0 < \alpha \text{ fixé } < 1) \end{cases}$$

$w(x_0)$ arbitrairement voisin de $\frac{1}{2}(1 - v_1(x_0))$, ce qui permet par un facteur de proportionnalité < 1, de le ramener égal à $\frac{1}{3}(1 - v_1(x_0))$.

Alors $v_1 + w$ est une fonction v_2 surharmonique continue, harmonique dans ω satisfaisant à :

$$\begin{aligned} & 0 < v_2 < 1 \\ & v_2(x_0) = v_1(x_0) + \tfrac{1}{3}(1 - v_1(x_0)) \\ & v_2(x) \leqq v_1(x) + \tfrac{3}{2}(v_2(x_0) - v_1(x_0)) \\ & v_2(x) \leqq v_1(x) + \alpha(1 - \sup_{C\delta} v_1) \quad \text{dans} \quad C\delta. \end{aligned}$$

On recommence à partir de v_2 et on répète l'opération en prenant des δ à rayon tendant vers 0 et des α formant les termes d'une série convergente.

5. D'après un théorème de Evans-Vasilesco (ou régularité du noyau de Green).

On voit que v_n toujours du même type, croît, converge uniformément et satisfait à :

$$0 < v_n < 1 \qquad v_n(x_0) \to 1$$

et en tout $x \neq x_0$, $v_n(x) < v_{n-1}(x) + \alpha_n(1 - v_{n-1}(x))$ à partir d'un rang N. D'où :

$$1 - v_n(x) > (1-\alpha_n)(1-\alpha_{n-1}) \ldots (1-\alpha_N)(1-v_{N-1}(x)) \to \lim_{n\to\infty}\text{ite} > 0\,.$$

Ainsi $\lim v_n(x)$ répond à la question.

III. LE THÉORÈME DE KELDYCH

4. Théorème 1. *Soit pour* ω *ouvert borné de* R^n, *la fonctionnelle* $\mathscr{L}_f(x)$ *croissante de* $f \in \mathscr{C}$, *pour tout* $x \in \omega$, *harmonique en* x *et égale à* H_f *pour toute* $f \in \mathscr{C}_R$. *Alors* $\mathscr{L}_f = H_f$.

Il est aussi simple d'utiliser le lemme affaibli.

ε étant donné, soit δ un voisinage du point-frontière régulier x_0, sur lequel $f < f(x_0) + \varepsilon$. Soit φ une fonction du Lemme 1, majorant $\sup f = f(x_0)$ hors δ. Alors $f \leqq f(x_0) + \varepsilon + \varphi$ sur ω^* d'où :

$$\mathscr{L}_f \leqq \mathscr{L}_{f(x_0)+\varepsilon+\varphi} = H_{f(x_0)+\varepsilon+\varphi}\,.$$

Donc :

$$\limsup_{x\in\omega,\, x\to x_0} \mathscr{L}_f(x) \leqq f(x_0) + \varepsilon + \varphi(x_0) \leqq f(x_0) + 2\varepsilon$$

$$\limsup_{x\in\omega,\, x\to x_0} \mathscr{L}_f \leqq f(x_0) \quad \text{et de même} \quad \liminf_{x\to x_0} \mathscr{L}_f \geqq f(x_0).$$

Ainsi $\mathscr{L}_f$ tend vers $f(x_0)$ en tout point-frontière régulier x_0. Or $\mathscr{L}_f$ est fonction bornée de x car

$$\inf f = \mathscr{L}_{\inf f} \leqq \mathscr{L}_f \leqq \mathscr{L}_{\sup f} = \sup f\,.$$

Donc $\mathscr{L}_f - H_f$ est une fonction harmonique bornée s'annulant aux points-frontière réguliers. Elle est nulle.

Si l'on dispose du lemme de Keldych, on peut prendre pour φ une fonction de ce lemme multipliée par un facteur convenable, ce qui donne $\varphi(x_0) = 0$, sans changer sensiblement la démonstration.

IV. RAPPEL D'UNE AXIOMATIQUE (6) ET COMPLÉMENTS

5. Soit Ω un espace topologique localement compact et connexe, non compact et rendu compact selon $\overline{\Omega}$ par adjonction du point d'Alexandroff. On adopte la topologie de $\overline{\Omega}$. On fait correspondre à chaque ouvert $\omega \subset \Omega$ un espace vectoriel réel de fonctions finies continues sur ω, dites harmoniques et on suppose satisfaits les axiomes suivants :

Axiome 1 (Axiome de faisceau). Toute fonction harmonique dans ω est harmonique dans toute partie ouverte et toute fonction harmonique dans un voisinage ouvert de chaque point de ω est harmonique dans ω.

Ouvert régulier. On dit que ω est régulier si $\overline{\omega} \subset \Omega$ et si pour toute fonction réelle finie continue sur la frontière ω^*, il existe une seule fonction harmonique dans ω se prolongeant continûment selon f et qui soit $\geqq 0$ si $f \geqq 0$. Cette "solution du problème classique pour ω et f" notée $H_f(x)$ est donc de la forme $\int f(y)\, d\rho_x^\omega(y)$ ($d\rho_x^\omega$ mesure de Radon sur ω^*, dite mesure harmonique relative à $x \in \omega$). On voit que si φ est bornée supérieurement

$$\limsup_{x \in \omega,\, x \to x_0 \in \omega^*} \overline{\int} \varphi\, d\rho_x^\omega \leq \limsup \varphi \quad \text{en} \quad x_0 .$$

Axiome 2 (Axiome de résolubilité locale du problème de Dirichlet). Il existe une base de domaines réguliers.

Axiome 3 (Axiome de convergence). Pour tout ensemble ordonné filtrant croissant de fonctions harmoniques u_i dans un domaine, $\sup u_i$ est harmonique ou vaut $+\infty$.

Cela équivaut à la même condition pour les suites croissantes si Ω est à base dénombrable ; et dans le cas général, lorsque 1 et 2 sont supposés, (3) équivaut à ce que pour tout domaine régulier, la sommabilité-$d\rho_x^\omega$ soit indépendante de $x \in \omega$ et l'intégrale $\int f d\rho_x$ pour une fonction f sommable-$d\rho_x$ quelconque soit continue de x.

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6. Axiomatique développée dans le Séminaire de théorie du potentiel ([4] et [5]), et dans un cours du Tata Institute [6].

Cette axiomatique inspirée de Doob ([8] et [9]) est prolongée par les travaux actuels de Mme Hervé ([10], [11], [12], [13]). Signalons des recherches similaires de Kamke [14] et H. Bauer [(1], [2] et [3]).

Exemples.

1° On prend pour Ω un "espace $\mathcal{E}$" non compact (voir [7]), en particulier R^n ou une surface de Riemann non compacte. On prend, comme fonctions harmoniques, celles qui, localement, deviennent sur l'espace-image R^n pourvu de son point à l'infini, harmoniques au sens classique (avec extension convenable au voisinage de ce point à l'infini).

2° Dans R^n (ou dans des variétés plus générales), on prend comme fonctions harmoniques, les solutions de l'équation de type elliptique à coefficients assez réguliers

$$\sum_{ij} a_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_i b_i \frac{\partial u}{\partial x_i} + cu = 0 \qquad c \leqq 0 \,.$$

Fonctions hyperharmoniques. Ce sont dans ω ouvert les fonctions u satisfaisant aux conditions :

$$\begin{cases} u \text{ semi-continue inférieurement, } u > -\infty \,, \\ u(x) \geqq \int u d\rho_x^\omega \text{ pour tout } \omega \text{ régulier} \end{cases}$$

(ou, ce qu'on démontre être équivalent, pour les domaines d'une base de domaines réguliers).

Dans un domaine, une telle fonction est $+\infty$ ou finie sur un ensemble partout dense. On dit dans ce dernier cas qu'elle est surharmonique.

On dit que u est hypo (sous)-harmonique si $-u$ est hyper (sur)-harmonique. Le remplacement de u par $\int u d\rho_x^\omega$ dans ω conserve l'hyper-harmonicité.

Lorsque les constantes sont harmoniques, les fonctions hyperharmoniques satisfont au "principe du minimum".

Lorsque u surharmonique admet une minorante sous-harmonique, il en existe une maxima qui est harmonique ; lorsque la plus grande minorante harmonique est nulle, u est appelé "potentiel $\geqq 0$" ou brièvement "potentiel". S'il existe un tel potentiel >0, il existe aussi un potentiel fini continu >0.

L'ordre "spécifique" des fonctions surharmoniques est défini par $v_1 \succ v_2$ signifiant :

$$v_1 = v_2 + \text{fonction surharmonique} \geqq 0 \,.$$

Rappelons enfin que s'il existe dans Ω une fonction harmonique $h>0$, les quotients par h des fonctions harmoniques (quotients qui comportent alors les constantes) satisfont aux axiomes avec mêmes ouverts réguliers

(ce sont les fonctions dites h-harmoniques) et que les fonctions hyperharmoniques correspondantes sont les quotients par h des anciennes.

Théorème de prolongement. Sous sa forme générale (donnée par Mme Hervé), on considère une fonction surharmonique $u \geqq 0$ dans ω et ω' ouvert quelconque $\subset \overline{\omega'} \subset \omega$. S'il existe un potentiel >0 dans Ω, il existe un potentiel $\geqq 0$ dans Ω qui dans ω', vaut u à une fonction harmonique près.

Théorème de partition (Mme Hervé). Pour tout ouvert ω, toute fonction surharmonique $\geqq 0$ dans Ω se décompose en une somme de deux fonctions surharmoniques $\geqq 0$, $v = v_1^{\omega} + v_2^{\omega}$, où v_1 est la plus grande minorante spécifique surharmonique de v, harmonique dans ω. Si Ω est à base dénombrable et $v(x_0)$ fini, $v_2^{\omega}(x_0)$ est une fonction de ω qui se prolonge en fonction d'ensemble ν_{x_0}, mesure de Radon sur Ω (mesure de Mme Hervé).

Le problème de Dirichlet. Soit ω ouvert $\subset \overline{\omega} \subset \Omega$ et l'ensemble $\mathscr{C}$ des fonctions réelles finies continues f sur ω^*. En supposant l'existence d'un potentiel >0 dans Ω, sans plus, Mme Hervé montre l'approximation à ε près de toute $f \in \mathscr{C}$ par une différence de deux potentiels $\geqq 0$ finis continus dans Ω; il s'ensuit que les deux enveloppes analogues à celles de Perron-Wiener sont égales et harmoniques. C'est la solution généralisée $H_f^{\omega}(x)$ qui s'écrit $\int f d\mu_x^{\omega}$ ($d\mu_x^{\omega}$ mesure de Radon sur ω^*, dite mesure harmonique et qui, si ω est régulier, vaut $\int f d\rho_x^{\omega}$.

On notera encore $\mathscr{C}_R^{\omega}$ ou $\mathscr{C}_R$ l'ensemble des f "classiquement résolutives" c'est-à-dire telles que H_f se prolonge continûment par f sur ω^*.

Un point-frontière est dit régulier si $H_f^{\omega} \underset{x \to x_0}{\rightarrow} f(x_0)$ quelle que soit $f \in \mathscr{C}$.

6. Ensembles polaires.

Un ensemble $e \subset \Omega$ est dit polaire s'il existe une fonction surharmonique >0 dans Ω, valant $+\infty$ au moins sur e. Quasi partout signifie "sauf sur un ensemble polaire" :

Lorsqu'il existe un potentiel >0 dans Ω, un ensemble localement polaire est polaire (voir [5] ou [6]). Les ensembles ouverts polaires du sous-space ω^* (ω ouvert $\subset \overline{\omega} \subset \Omega$) sont les ouverts de ω^* de $d\mu_x$-mesure nulle quel que soit $x \in \omega$: il est d'abord facile de voir que tout ensemble polaire de ω^* est de $d\mu_x$-mesure nulle. Réciproquement, supposons que e ouvert sur ω^* soit de $d\mu_x$-mesure nulle quel que soit x, soit δ un domaine

ne coupant ω^* que selon une partie de e et δ' un domaine composant de $\delta \cap \omega$; la partie de frontière e' située dans δ appartient à e et est de mesure harmonique nulle pour δ' comme pour ω d'où l'existence d'une fonction surharmonique $\geqq 0$ dans δ', tendant vers $+\infty$ aux points de e'; prolongée par $+\infty$, cette fonction donnerait une fonction hyperharmonique $\geqq 0$ dans δ, ce qui exige $\delta \cap \omega$ connexe et $e \cap \delta$ polaire dans δ donc dans Ω.

Il existe alors sur ω^* un ouvert polaire maximum (réunion des ouverts polaires) qu'on appellera *partie impropre* de ω^*. Le reste sera dit partie propre de ω^*. On voit que tout point de la partie impropre est irrégulier.

On sait que si dans un domaine δ, un ensemble e fermé dans δ est polaire, $\delta - e$ est connexe et toute fonction surharmonique (resp. harmonique bornée) dans $\delta - e$ se prolonge surharmoniquement (harmoniquement) de façon unique dans δ.

Enfin, rappelons que, selon Mme Hervé, pour la mesure ν_{x_0} associée comme plus haut à une fonction surharmonique $\geqq 0$ localement bornée (lorsque Ω est à base dénombrable) tout ensemble polaire est de ν_{x_0}-mesure nulle.

7. L'axiome D.

Nous aurons besoin d'une propriété équivalente à l'axiome suivant, introduit dans un développement plus avancé de l'axiomatique précédente:

Axiome D (ou principe de domination). *Si v est un potentiel localement borné $\geqq 0$ dans Ω, toute fonction surharmonique $\geqq 0$ majorant v sur son support (complementaire de l'ensemble ouvert maximum où v est harmonique) majore v partout.*

Conséquences des axiomes 1, 2, 3, D.

1° Rappelons que pour toute fonction u surharmonique localement bornée, si la restriction à son support est continue en un point, u est continue en ce point dans Ω.

Remarque (Mme Hervé). En supposant les axiomes 1, 2, 3 et Ω à base dénombrable, l'axiome D équivaut à ce que pour toute fonction v surharmonique localement bornée, la continuité de la restriction sur le support entraîne la continuité dans Ω.

2° Une application fondamentale de (D) jointe aux axiomes 1, 2, 3

et à l'hypothèse de Ω à base dénombrable, est que dans le problème de Dirichlet considéré plus haut pour ω (dans l'hypothèse d'existence d'un potentiel >0) l'ensemble des points-frontière irréguliers est polaire, donc la partie propre de la frontière est l'adhérence de l'ensemble des points réguliers. Toute fonction harmonique et bornée dans ω s'annulant aux points-frontière réguliers est nulle.

V. EXTENSION DU THÉORÈME DE KELDYCH DANS LE CAS GÉNÉRAL

8. Il est commode de traiter d'abord le cas où les constantes sont harmoniques.

Lemme 2. *On suppose Ω à base dénombrable, les axiomes 1, 2, 3, D, l'existence d'un potentiel >0, et les constantes harmoniques. Soit ω ouvert $\subset\overline{\omega}\subset\Omega$ et x_0 un point-frontière régulier formant un ensemble polaire. Alors l'énoncé du Lemme 1 est vrai (en prenant pour ω_0 un domaine tel que $\overline{\omega}\subset\omega_0\subset\overline{\omega}_0\subset\Omega$) dans la théorie axiomatique précédente.*

La démonstration s'adapte par exemple comme suit :

On introduit ω_0 comme indiqué et un voisinage b de x_0. La solution pour $\omega-\overline{b}$, la donnée K' sur ω_0^* et 0 sur b^* tend vers K' selon le filtre de b.

Elle a en effet une limite harmonique v hors x_0 donc se prolongeant harmoniquement dans ω_0 et comme v est bornée et de limite K' aux points réguliers de ω_0, elle vaut K'. La convergence est uniforme hors d'un voisinage de x_0.

On choisira donc σ et δ comme au numéro 2, mais le prolongement de la solution par 0 dans $\overline{b}$ n'est peut-être pas continu et sous-harmonique ; on prendra la régularisée (semi-continue supérieurement et sous-harmonique) puis on en déduit une fonction finie continue sous-harmonique et majorante, mais de même valeur hors d'un voisinage de b^* (on utilise des ω_i réguliers couvrant b^* et le passage d'une fonction sous-harmonique w à $\int dw\rho_x^{\omega_i}$ opéré successivement ; voir [5] ou [6]). A partir de cette fonction U, on passe à u en remplaçant U par H_U^{ω} dans ω et la régularisation semi-continue donne $\widehat{u}$ sous-harmonique mais harmonique dans w et satisfaisant à

$$\widehat{u}\geqq 0 \quad , \quad \widehat{u}\leqq K' \quad , \quad \widehat{u}(x_0)=0 \quad , \quad \widehat{u}\geqq K \quad \text{hors } \delta .$$

On considère alors $K' - \widehat{u} = v$ surharmonique $\geqq 0$ dans ω_0 et, selon le théorème de partition appliqué à ω_0, la mesure ν_{x_0} associée qui est d'ailleurs supportée par $\omega^* \cup b^*$. Il existe un ouvert σ de ν_{x_0}-mesure $< \alpha$ tel que la restriction de v sur $\complement\sigma$ soit continue ; la composante v_1^σ est aussi de restriction continue sur $\complement\sigma$ et v_1^σ est donc continue dans ω_0, harmonique dans ω. Enfin $v(x_0) - v_1^\sigma(x_0) < \alpha$.

On peut aussi, comme dans le cas classique, donner une autre démonstration en prenant σ contenant les points irréguliers et non x_0.

Lemme 3. Soit, avec les mêmes hypothèses qu'au Lemme 2, un point-frontière régulier x_0 de ω, tel que $\{x_0\}$ soit non polaire. Soit ω_0 un domaine contenant $\overline{\omega}$, δ un voisinage ouvert de x_0 ($\overline{\delta} \subset \omega_0$). Il existe sur ω^* une fonction $\vartheta \in \mathscr{C}_R^\omega$ satisfaisant à $\vartheta \geqq 0$, $\vartheta(x_0) = 0$, $\vartheta \geqq K$ donné hors δ, quasi-partout (c'est-à-dire sur la partie propre de ω^*). On la réalise en définissant une fonction surharmonique $V \geqq 0$ dans $\omega_0 - \{x_0\}$, finie continue, s'annulant en x_0, harmonique hors d'un compact non polaire de ω^* et majorant K quassi-partout hors δ.

Considérons les domaines composant $\omega_0 - \{x_0\}$ et retenons ceux ω_i, en nombre fini, qui rencontrent δ^* et coupent ω^* selon un ensemble non polaire, c'est-à-dire contiennent un compact non polaire K_i de ω^*. On introduit $H_\varphi^{\omega_i - K_i}$ où φ continue vaut 0 en x_0 et 1 ailleurs. On prolonge cette fonction sur K_i, par les valeurs de la limite inférieure aux points de K_i, qui valent 1 aux points réguliers pour $\omega_i - K_i$ (il en existe puisque K_i est non polaire). On obtient ainsi une fonction surharmonique $v_i > 0$ dans ω_i. Grâce au théorème de partition, on en déduit une autre $v'_i \leqq v_i$, donc tendant vers 0 en x_0, continue dans ω_i, harmonique dans $\omega_i - K_i$ et arbitrairement voisine de v_i en un point fixé, donc > 0 dans ω_i. Comme $\omega_i \cap (\delta^* \cup \omega^*)$ est compact, $\lambda_i v'_i$ pour λ_i convenable majore K fixé > 0 sur ce compact, donc sur l'ouvert $\complement\overline{\delta} \cap \omega \cap \omega_i$ (dont la frontière appartient au compact précédent). On considère alors dans $\omega_0 - \{x_0\}$ la fonction égale à $\lambda_i v'_i$ dans chaque ω_i et à 0 dans les autres domaines composants. C'est une fonction V de l'énoncé qui fournit ϑ cherchée.

Lemme 4. On améliore le Lemme 3 dans les mêmes hypothèses comme suit (sans introduire de voisinage δ

ni de minorante K): il existe sur ω^* une fonction $\vartheta \in \mathscr{C}_R^{\omega}$ satisfaisant à $\vartheta(x_0) = 0$, $\vartheta(x) > 0$ sur $\omega^* - \{x_0\}$ quasi-partout (c'est-à-dire hors de la partie impropre de ω^*). De plus, si $\Omega - \{x_0\}$ est connexe, ce "quasi-partout" peut être remplacé par "partout".

Ce dernier cas se traite immédiatement. On peut choisir un domaine $\omega_0 \subset \overline{\omega_0} \subset \Omega$ contenant $\overline{\omega}$ et tel que $\omega_0 - \{x_0\}$ soit connexe. La fonction $H_{\varphi}^{\omega_0 - \{x_0\}}$ où φ vaut 1 sur ω_0^* et 0 en x_0 est > 0 parce qu'il y a au moins un point-frontière régulier de $\omega_0 - \{x_0\}$ sur ω_0^* (sinon ω_0^* serait polaire, donc de mesure harmonique nulle pour ω_0); elle tend vers zéro au point x_0 (régulier pour $\omega_0 - \{x_0\}$). Donc sa trace sur ω^*, prolongée par 0 en x_0 est une ϑ répondant à la question.

Traitons le cas général en reprenant la démonstration du Lemme 3 et en introduisant une suite décroissante de voisinages ouverts δ_n de x_0 tels que $\cap \delta_n = \{x_0\}$. A δ_1 associons la réunion Ω_1 des domaines composants de $\omega_0 - \{x_0\}$ tels qu'ils rencontrent δ_1^* et coupent ω^* selon un ensemble non polaire; il existe, d'après la démonstration précédente, une fonction surharmonique v_1 dans Ω_1, continue > 0, ≤ 1, harmonique dans $\omega \cap \Omega_1$, et tendant vers 0 en x_0.

A δ_2 associons les domaines composants de $\omega_0 - \{x_0\}$ qui sont contenus dans δ_1 mais coupent δ_2^* et rencontrent ω^* selon un ensemble non polaire; sur leur réunion Ω_2, on peut définir une fonction surharmonique $v_2 > 0$, continue, $\leq \frac{1}{2}$, harmonique dans $\omega \cap \Omega_2$, tendant vers 0 en x_0; et ainsi de suite, v_n étant $\leq \frac{1}{n}$.

Sur la réunion des Ω_n est ainsi définie une fonction qu'on prolonge par 0 dans les domaines de $\omega_0 - \{x_0\}$ qui coupent ω^* suivant un ensemble polaire. La fonction obtenue prolongée par 0 en x_0 donne sur ω^* une fonction ϑ répondant à la question.

9. Extension.

En supprimant la restriction que les constantes sont harmoniques, les Lemmes 3 et 4 subsistent tels quels et aussi le Lemme 1, 2 en supprimant à la fin le choix arbitraire de $K' > K$. Il suffit d'introduire un domaine Ω' tel que $\overline{\omega_0} \subset \Omega' \subset \overline{\Omega'} \subset \Omega$, et une fonction harmonique $h > 0$ dans Ω'. On considère dans Ω' les mêmes problèmes à partir des fonctions h-

harmoniques ; on applique les lemmes précédents et on utilise le fait que h est sur ω_0 compris entre deux nombres >0.

Concluons : le Lemme 4 devient le :

Lemme de Keldych étendu 5: On suppose Ω à base dénombrable, les axiomes 1, 2, 3, D et l'existence d'un potentiel >0. Si x_0 est un point-frontière régulier de $\omega \subset \overline{\omega} \subset \Omega$, il existe sur ω^* une fonction $\vartheta \in \mathscr{C}_R^{\omega}$, telle que $\vartheta(x_0) = 0$, $\vartheta(x) > 0$ pour $x \neq x_0$ sur ω^* seulement quasi-partout (ou sur la partie propre, hors de x_0). Si $\Omega - (x_0)$ est connexe (en particulier si $\{x_0\}$ est polaire), "quasi-partout" peut être remplacé par "partout".

10. Théorème de Keldych étendu 2.

Dans les hypothèses du lemme précédent, soit $\mathscr{L}_f(x)$ réel fini satisfaisant aux conditions suivantes:

1° $\mathscr{L}_f(x)$ est fonction harmonique de x pour chaque $f \in \mathscr{C}^{\omega}$.

2° $\mathscr{L}_f(x) = H_f(x)$ pour toute $f \in \mathscr{C}_R^{\omega}$.

3° $\mathscr{L}_f(x)$ est pour chaque x fonction croissante de f.

4° $\mathscr{L}_{f_1} = \mathscr{L}_{f_2}$ si $f_1 = f_2$ quasi-partout, ou, ce qui est équivalent, $\mathscr{L}_f$ ne dépend que des valeurs de f sur la partie propre de ω^*.

Alors $\mathscr{L}_f = H_f$, solution généralisée pour toute $f \in \mathscr{C}^{\omega}$.

Cela résulte du lemme de Keldych étendu ou seulement de l'ensemble plus faible des Lemmes 2 et 3. Même raisonnement que dans le cas classique (numéro 4), sauf qu'au début on écrira :

$$f \leqq f(x_0) + \varepsilon + \varphi \quad \text{quasi-partout.}$$

Si f_1 est le second membre, $\inf(f, f_1)$ vaut f quasi-partout, d'où :

$$\mathscr{L}_f = \mathscr{L}_{\inf(f, f_1)} \leqq \mathscr{L}_{f_1} = H_{f(x_0)+\varepsilon+\varphi}.$$

Même fin du raisonnement.

Remarque 1. La condition (4) disparait s'il n'y a pas de partie impropre, pas de point non polaire dans Ω ou si pour tout point-frontière non polaire x_0, $\Omega - \{x_0\}$ est connexe.

Remarque 2. (Contre-exemple). La condition (4) est nécessaire d'après l'exemple suivant : il suffit de réaliser un espace Ω dont un domaine $\sigma \neq \Omega$ n'admet dans $\overline{\Omega}$ qu'un seul point-frontière, le point non polaire $x_0 \in \Omega$ et contient un point polaire x_1, puis de choisir des fonctions harmoniques satisfaisant aux axiomes du théorème précédent et comportant les constantes.

Car si ω est l'ouvert σ diminué du point x_1, prenons pour $\mathscr{L}_f(x)$ la constante $\frac{1}{2}[f(x_0)+f(x_1)]$. H_f^ω vaut la constante $f(x_0)$ et f n'est classiquement résolutive que si $f(x_0) = f(x_1)$. Les conditions 1, 2, 3 sont satisfaites, mais l'on n'a pas toujours $\mathscr{L}_f = H_f$.

Pour réaliser l'espace Ω et les fonctions harmoniques considérées, on prendra un ensemble E_1 formé d'un espace R_{n_1} ($n_1 \geqq 3$) diminué d'une boule fermée σ mais pourvu de son point à l'infini A, et un ensemble E_2 formé d'un espace R_{n_2} ($n_2 \geqq 3$) pourvu de son point à l'infini qui sera identifié à A. La topologie de la réunion Ω de ces ensembles sera définie par les voisinages ; pour $x \neq A$, une base sera celle des voisinages dans E_1 ou E_2 ; si $x = A$, on réunit un voisinage quelconque de E_1 et un voisinage quelconque de E_2 pour avoir un voisinage quelconque de A. Une fonction finie continue sera harmonique dans un ouvert de $E_1 - A$ ou $E_2 - A$ si elle l'est dans R_{n_1} ou R_{n_2} ; elle sera harmonique dans un ouvert $\omega \subset \Omega$ contenant A si elle l'est dans $\omega \cap E_1 - \{A\}$ et $\omega \cap E_2 - \{A\}$ et si la somme des flux en A calculés dans R_{n_1} et R_{n_2} et même, multipliés par des facteurs >0 fixés, est nulle.[7]

Grâce à la théorie classique dans l'espace euclidien pourvu du point à l'infini, on vérifie les divers axiomes et conditions nécessaires et d'abord les axiomes 1, 2, 3 (et même 3'). Evidemment une fonction surharmonique dans $\omega \subset \Omega$ est dans $R_{n_1} \cap \omega$, $R_{n_2} \cap \omega$ aussi surharmonique au sens relatif aux espaces euclidiens R_{n_1} et R_{n_2}. De plus, toute fonction surharmonique u dans Ω reste surharmonique quand on la remplace dans R_{n_2} par la constante $u(A)$ et on en déduit que u est surharmonique dans E_1 (au sens relatif aux espaces euclidiens) ; par suite, grâce au théorème du prolongement, si une fonction surharmonique au voisinage de A dans Ω est finie continue sur son support en A, elle est continue sur E_1 au point A ; comme on

7. On adapte ici une idée de Mme Hervé. Le flux en A dans R_n est la limite (quand le rayon tend vers l'infini) du flux entrant dans une boule de centre fixe (quelconque) de R_n.

peut permuter le rôle de R_{n_1} et R_{n_2}, u est aussi continue sur E_2 en A. Il s'ensuit la validité de l'axiome D.

VI. PROBLÈMES ANALOGUES DE PROLONGEMENT FONCTIONNEL

11. Prenons le cas de l'axiomatique générale et les hypothèses générales du théorème de Keldych étendu (axiomes 1, 2, 3, D, Ω à base dénombrable, existence d'un potentiel >0). On notera en outre $\mathscr{C}^+$, $\mathscr{C}_R^+$ les parties de $\mathscr{C}$, $\mathscr{C}_R$ formées de fonctions $\geqq 0$. On considèrera encore pour $f \in \mathscr{C}$, $\mathscr{L}_f(x)$ satisfaisant aux conditions 1 et 2 du théorème. Nous voudrions conclure de même, mais au lieu d'imposer directement la condition de croissance (3) qui est une propriété de la solution généralisée H_f, on songera à imposer à $\mathscr{L}_f$ d'autres propriétés de H_f, (4) devant ou non être maintenue :

Proposition 1. Soit T une topologie sur $\mathscr{C}$ pour laquelle $\mathscr{C}_R^+$ est partout dense dans $\mathscr{C}^+$. On peut, dans le Théorème 2, remplacer la condition de croissance (3) par celle que $\mathscr{L}_f(x)$ est linéaire et continue de $f \in \mathscr{C}$ pour chaque x.

Car $\mathscr{L}_f$ sera $\geqq 0$ si $f \geqq 0$, donc $\mathscr{L}_f$ sera croissante.

Proposition 2. Soit T_1 une topologie sur $\mathscr{C}$ pour laquelle $\mathscr{C}_R$ est partout dense dans $\mathscr{C}$ et la solution générale $H_f(x)$ continue de f sur $\mathscr{L}$. Alors, dans le Théorème 2, on peut remplacer les conditions 3 et 4 par celle que $\mathscr{L}_f(x)$ est continue de $f \in \mathscr{C}$ pour chaque x.

Car cette continuité et l'identité à H_f sur $\mathscr{C}_R$ entraîne l'identité sur $\mathscr{C}$. On est donc amené à rechercher les topologies T et T_1 précédentes. En voici des exemples :

Théorème 3. Considérons la topologie $\mathscr{T}$ (sur $\mathscr{C}$) des semi-normes $\int |f|\, d\mu_x$ (pour tous les $x \in \omega$, ce qui équivaut, si l'axiome (3') (voir [5], [6]) est satisfait, à prendre un x dans chaque domaine composant); elle est plus fine que celle $\mathscr{T}_1$ des semi-normes $\left|\int f d\mu_x\right|$, laquelle est la moins fine rendant $H_f(x)$ continue de f, quel que soit x.

$\mathscr{T}$ donc $\mathscr{T}_1$ rendent $\mathscr{C}_R^+$ partout dense dans $\mathscr{C}^+$ (donc $\mathscr{C}_R$ partout dense dans $\mathscr{C}$).

La première partie est banale. Il s'agit ensuite de montrer qu'étant

donné $f \in \mathscr{C}^+$, $x_1, x_2 \dots x_p$ dans ω, $\varepsilon > 0$, il existe $\varphi \in \mathscr{C}^+_R$ telle que

$$\int |f - \varphi| \, d\mu_{x_i} < \varepsilon \qquad (i = 1, 2, \dots, p).$$

On sait qu'on peut approcher f sur ω^* uniformément arbitrairement par une différence de deux potentiels $\geqq 0$. Si W est un potentiel fini continu >0 dans Ω, on voit d'abord qu'on peut trouver deux potentiels $\geqq 0$, V_1, V_2 et $\alpha > 0$ tels que

$$\begin{cases} V_1 - V_2 \geqq -\alpha W \quad \text{sur } \omega^* \\ \int |V_1 - V_2 - f| \, d\mu_{x_i} < \dfrac{\varepsilon}{8} \\ \alpha \int W \, d\mu_{x_i} < \dfrac{\varepsilon}{8} \end{cases}$$

donc que $W_1 = V_1 + \alpha W$ et $W_2 = V_2$ sont deux potentiels $\geqq 0$ satisfaisant à

$$W_1 \geqq W_2 \quad \text{sur } \omega^*$$

$$\int |f - (W_1 - W_2)| \, d\mu_{x_i} < \frac{\varepsilon}{4} \qquad (i = 1, 2, \dots, p).$$

On remplace W_1 par $H^{\omega}_{W_1}$ dans ω d'où une fonction dont la régularisée semi-continue inférieurement est un potentiel W'_1, harmonique dans ω où elle vaut $\int W_1 \, d\mu_x$. On en déduit, en utilisant comme plus haut le théorème de partition et la mesure de Mme Hervé (soit avec la propriété de Lusin, soit avec un ouvert contenant les points irréguliers) un potentiel $W''_1 \leqq W'_1 \leqq W_1$, continu dans Ω, harmonique dans ω et satisfaisant à :

$$\int (W_1 - W''_1) \, d\mu_{x_i} < \frac{\varepsilon}{4}.$$

De même, on introduit W'_2, puis W''_2 $(0 \leqq W''_2 \leqq W_2)$ continu, harmonique dans ω et satisfaisant à

$$\int (W_2 - W''_2) \, d\mu_{x_i} < \frac{\varepsilon}{4}.$$

Sur ω^*, on sait que $W'_1 \geqq W'_2 \geqq W''_2$ mais on ne sait si $W''_1 \geqq W''_2$. Aussi allons-nous introduire au lieu de W''_1 une suite croissante Π_n de potentiels de ce même type, et de limite W'_1, par exemple au moyen d'une suite décroissante d'ouverts dont l'intersection est l'ensemble des points irréguliers. Alors, pour n choisi assez grand

$$\Pi_n \geqq W''_2 - \alpha W \quad \text{sur } \omega^*,$$

tandis que

$$\int |f - (\Pi_n + \alpha W - W_2'')| \, d\mu_{x_i} < \varepsilon .$$

La trace sur ω^* de $\Pi_n + \alpha W - W_2''$ est une φ répondant à la question.

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RELATIVE LIMIT THEOREMS IN ANALYSIS

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1. Introduction.

In many fields of analysis, for example in the theory of Fourier series and in the theory of harmonic functions, there are classical limit theorems, dating back to the first decade after the introduction of measure theory into analysis, which state roughly that, when a measure U of Borel sets determines a sequence of numbers in a certain way, then the sequence converges to U' (derivative with respect to Lebesgue measure) at a specified point. An early example is Lebesgue's theorem that the Fourier series of a summable function f converges $(C, 1)$ almost everywhere to the function. In this case U is the measure having density f with respect to Lebesgue measure. Such theorems led in a natural way to sequences defined in terms of kernels,

$$(1.1) \qquad u_n(\xi) = \int K(n, \xi, \eta)\, U(d\eta),$$

for which it can be proved that, under suitable restrictions on the kernel,

$$(1.2) \qquad \lim_{n \to \infty} u_n(\xi) = U'(\xi)$$

for almost all ξ [see for example VIII p. 28].

More recently [I, II] examples have been discovered in which, if u_n is given by (1 . 1) in terms of a measure U, and if h_n is given similarly in terms of a measure H using the same kernel, (1.2) can be generalized to

$$(1.3) \qquad \lim_{n \to \infty} \frac{u_n(\xi)}{h_n(\xi)} = \frac{dU}{dH}(\xi)$$

for H-almost every ξ. The purpose of this paper is to derive conditions on kernels for which this generalization is true. Applications are made to harmonic and analogous functions, as well as to Fourier series. An alternative approach given in [II] is more clumsy and more difficult to apply.

2. Set functions and derivatives.

Let U be a signed measure of Borel subsets of Euclidean N-space, finite for compact sets. The absolute variation $\|U\|$ is then a measure, also finite for compact sets. A Baire function f will be called summable with respect to U if and only if $|f|$ is summable with respect to $\|U\|$.

If $N = 1$ and if U and H are two such signed measures, dU/dH will denote the derivative, defined at a point ξ by the following limit

$$\lim_{|I_\xi| \to 0} \frac{U(I_\xi)}{H(I_\xi)} = \frac{dU}{dH}(\xi), \tag{2.1}$$

if the denominator on the left is not zero for sufficiently small $|I_\xi|$, and if the indicated limit exists. Here I_ξ is any closed interval containing the point ξ and $|I_\xi|$ is the length. If H is Lebesgue measure, we shall write U' for dU/dH. The limit in (2.1) exists and is finite at $\|H\|$ almost every point ξ [XI], and is the Radon-Nikodym derivative of U with respect to H. The reference only treats the case of positive (by which we always mean $\geqq 0$) H, but the general case is easily reduced to the positive one by the following remark. If A is a set of positivity of H, that is a Borel set on whose Borel subsets H is positive and on the Borel subsets of whose complement $-H$ is positive, then, aside from a set of $\|H\|$ measure 0,

$$\begin{aligned} \frac{dH}{d\|H\|}(\xi) &= 1 \quad \text{if } \xi \in A \\ &= -1 \quad \text{if } \xi \notin A, \end{aligned} \tag{2.2}$$

so that

$$\begin{aligned} \frac{dU}{dH}(\xi) &= \frac{dU}{d\|H\|}(\xi) \quad \text{if } \xi \in A \\ &= -\frac{dU}{d\|H\|}(\xi) \quad \text{if } \xi \notin A. \end{aligned} \tag{2.3}$$

This evaluation together with the fact that U is the difference between two measures enables one to reduce most proofs involving derivatives dU/dH to the case in which both U and H are positive.

If the dimensionality is greater than 1, the existence of the set function derivatives is covered by a terminological blanket, and the scientist who clarifies without addivelating the situation will deserve a perfect starred

halo. Dehubbing a definition which is unnecessarily general for our purposes, we shall say that U has the derivative a with respect to H if the following condition is satisfied. If S is a closed convex set containing ξ, let $d'(S)$ be the distance from ξ to the boundary of S, and let $d''(S)$ be the diameter of S. Then the condition is that, whenever $\{S_n, n \geqq 1\}$ is a sequence of closed convex sets containing ξ, for which

$$\lim_{n\to\infty} d''(S_n) = 0, \qquad \liminf_{n\to\infty} \frac{d'(S_n)}{d''(S_n)} > 0, \tag{2.4}$$

it follows that $H(S_n) \neq 0$ for large n, and that

$$\lim_{n\to\infty} \frac{U(S_n)}{H(S_n)} = a. \tag{2.5}$$

This derivative will be denoted by $(dU/dH)(\xi)$, or by $U'(\xi)$ if H is Lebesgue N-dimensional measure. The derivative dU/dH exists and is finite $||H||$ almost everywhere, and is the Radon-Nikodym derivative of U with respect to H [VI].

Suppose that $N \geqq 1$ and that there is a number a for which

$$\frac{d\,||U - aH||}{d\,||H||}(\xi) = 0. \tag{2.6}$$

Then $(dU/dH)(\xi) = a$, and we shall say that dU/dH has the value a at ξ in the variational sense. If $H(\{\xi\}) \neq 0$, dU/dH has the value $U(\{\xi\})/H(\{\xi\})$ in the variational sense.

Lemma 2.1. *The derivative dU/dH exists $||H||$ almost everywhere in the variational sense.*

This lemma is well known if $N = 1$ and if H is Lebesgue measure, going back to Lebesgue in that case if also U is absolutely continuous relative to H. The standard proof [XII, p. 59] is easily extended to the present case, and therefore will only be outlined. In the first place, if we denote dU/dH by f, and if a is any real number,

$$\frac{d\,||U - aH||}{d\,||H||} = |f - a| \tag{2.7}$$

$||H||$ almost everywhere. Then (2.7) is true $||H||$ almost everywhere simultaneously for all rational a, and it follows that (2.7) is true $||H||$ almost

everywhere simultaneously for all a. Finally, if ξ is not in the exceptional set for the last statement, we set $a = f(\xi)$ and deduce (2.6).

3. A general relative limit theorem.

In the following discussion, $K(n, .)$ will be a specified bounded positive Baire function on N-space, for each positive integer n. The rephrasing of the results to apply to an arbitrary open subset of N-space will be too obvious to require an explicit statement. To apply the results to a compact subset A of N-space, define $K(n, .)$ as 0 off the subset, or, which amounts to the same thing, consider only signed measures vanishing on the Borel subsets of the complement of A.

If U is a signed measure of Borel sets, finite for compact sets, define u_n by

$$u_n = \int K(n, \eta)\, U(d\eta). \tag{3.1}$$

Similarly, define h_n in terms of signed measure H. Let $\Gamma(K)$ be the class of signed measures U for which $K(n, .)$ is summable with respect to U for all n. We shall restrict our attention to signed measures in this class, which includes the signed measures of bounded variation, that is, the finite-valued signed measures.

Theorem 3.1. *Let U and H be in the class $\Gamma(K)$. Suppose that $(dU/dH)(0)$ and $(dH/d\|H\|)(0)$ exist in the variational sense, and that the latter derivative is either 1 or -1. Suppose also that $\|H\|'(0)$ exists, with $0 < \|H\|'(0) \leq \infty$. Suppose that $0 \leq K \leq 1$. Let c be a constant, with $c = 1$ if $N = 1$ but $0 < c < 1$ otherwise. Let $\{\delta_n, n \geq 1\}$ be a sequence of strictly positive numbers, and let $\{\eta_n, n \geq 1\}$ be a sequence of points in N-space. Suppose that the following conditions are satisfied.*

(a) $|\eta_n| \leq c\delta_n, \quad \lim_{n\to\infty} \delta_n = 0.$

(b) $\liminf_{n\to\infty} \inf_{|\eta-\eta_n|\leq\delta_n} K(n, \eta) > 0.$

(c) *For every $\alpha > 0$,*

$$K_1(n, \alpha) = \sup_{|\eta-\eta_n|\geqq\alpha} K(n, \eta) = o(\delta_n^N).$$

(d) There is a constant b for which

$$K_1(n, |\eta - \eta_n|) \leqq bK(n, \eta)$$

for all n, if $\delta_n \leqq |\eta - \eta_n| \leqq \alpha$ and if α is sufficiently small.

(e) If $\alpha > 0$, there is an integer $n_0 = n_0(\alpha)$, for which

$$K(n, \eta) \leqq o(\delta_n^N) K(n_0, \eta) \quad \text{when} \quad |\eta| \geqq \alpha.$$

Here $o(\delta_n^N)$ may depend on α but not on η.

Then, defining u_n, h_n as above,

$$\lim_{n\to\infty} \frac{u_n}{h_n} = \frac{dU}{dH}(0). \tag{3.2}$$

If U and H are of bounded variation, (e) is unnecessary.

Suppose that this theorem is known to be true when H is positive. We shall now show that it then must be true in the general case. Suppose for definiteness that $(dH/d\|H\|)(0) = 1$ and suppose that $(dU/dH)(0) = a$. Then under the hypothesis of the theorem, $(dU/d\|H\|)(0) = a$ in the variational sense, because, taking $N = 1$ for example, if I_0 is a closed interval containing 0,

$$\|(U-a\|H\|)\|(I_0) \leqq \|U-aH\|(I_0) + |a|\,\|(H-\|H\|)\|(I_0) = o(\|H\|(I_0)) \qquad (|I_0| \to 0). \tag{3.3}$$

Hence, if $\|h\|_n$ is defined like h_n but with H replaced by $\|H\|$, the application of the theorem to $u_n/\|h\|_n$ and $h_n/\|h\|_n$ yields

$$\lim_{n\to\infty} \frac{u_n}{\|h\|_n} = a, \qquad \lim_{n\to\infty} \frac{h_n}{\|h\|_n} = 1. \tag{3.4}$$

We conclude from (3.4) that (3.2) is true. Thus it is sufficient to prove the theorem under the hypothesis that H is positive, and we shall assume that this hypothesis is verified from now on.

In the following, we shall denote the closed ball of center η_n and radius r by $S_n(r)$. Define $V = \|U - aH\|$. We shall choose a strictly positive α below. It will be supposed throughout that α is so small that (d) is applicable, and that n is so large that $h_n > 0$ and $2\delta_n < \alpha$. With these conventions,

$$(3.5)\qquad \frac{|u_n - ah_n|}{h_n} \leqq \int_{S_n(\delta_n)} K(n, \eta)\, V(d\eta)/h_n + \int_{(\delta_n, \alpha]} K_1(n, r)\, dV(S_n(r))/h_n$$

$$+ \int_{(\alpha, \infty)} K_1(nr)\, dV(S_n(r))/h_n = I_1 + I_2 + I_3 .$$

Now h_n satisfies the inequality

$$(3.6)\qquad h_n \geqq H(S_n(\delta_n)) \inf_{\eta \in S_n(\delta_n)} K(n, \eta) .$$

Hence, when $n \to \infty$

$$(3.7)\qquad \begin{aligned} V(S_n(\delta_n))/h_n &= o(1), & H(S_n(\delta_n))/h_n &= O(1), \\ K_1(n, \alpha)/h_n &= o(1), & \delta_n^N/h_n &= O(1) . \end{aligned}$$

Applying (3.7), we find that

$$(3.8)\qquad I_1 \leqq V(S_n(\delta_n))/h_n = o(1) .$$

Integrating by parts,

$$(3.9)\qquad I_2 \leqq K_1(n, \alpha)\, V(S_n(\alpha))/h_n - \int_{(\delta_n, \alpha)} V(S_n(r))\, dK_1(n, r)/h_n .$$

Here the first term on the right is $o(1)$ by (3.7). If $\varepsilon > 0$ there is, since $(dV/dH)(0) = 0$, a value of α so small that

$$(3.10)\qquad V(S_n(r)) \leqq \varepsilon H(S_n(r)), \quad \text{if} \quad \delta_n \leqq r \leqq \alpha .$$

With this value of α, we find that the right side of (3.9) is at most

$$(3.11)\qquad o(1) - \varepsilon \int_{(\delta_n, \alpha]} H(S_n(r))\, dK_1(n, r)/h_n .$$

Integrating by parts again, this expression becomes

$$(3.12)\qquad o(1) + \varepsilon O(1) + \varepsilon \int_{(\delta_n, \alpha]} K_1(n, r)\, dH(S_n(r))/h_n \leqq o(1) + \varepsilon O(1) + b\varepsilon ,$$

where $O(1)$ does not depend on the choice of α. We have now proved that I_2 can be made arbitrarily small by choosing α sufficiently small. If U and H are both of bounded variation,

$$(3.13)\qquad I_3 \leqq K_1(n, \alpha)\, O(1)/h_n = o(1),$$

by (3.7). The left side of (3.5) is therefore $o(1)$, and we have thus

finished the proof of theorem, under the special hypothesis that U and H are of bounded variation.

If U and H are merely supposed to be in the class $\Gamma(K)$, the above discussion remains valid except for the treatment of I_3. Moreover, if (e) is true,

$$I_3 \leqq o(\delta_n^N) \int K(n_0, \eta) V(d\eta)/h_n = o(\delta_n^N)/h_n = o(1), \tag{3.14}$$

by (3.7), and we have thus proved that the left side of (3.5) is $o(1)$ in all cases.

If $K(n, .)$ vanishes strictly on one side of a hyperplane through η_n, the conditions of Theorem 3.1 cannot be satisfied, but the idea of the proof remains applicable if we simply ignore the half-spaces on which K vanishes. In this way we obtain the following theorem, which can be generalized to cover other zero-sets for K.

Theorem 3.2. *Theorem 3.1 remains valid if for each* n $K(n, .)$ *vanishes strictly on one side of a hyperplane through* η_n, *if in conditions* (b)—(e) η *is restricted to be in the complementary closed half-space* A_n, *if the origin lies in* A_n, *and if when* $N > 1$ *the origin is at distance* $\geqq (1-c)\delta_n$ *from the boundary of* A_n.

4. Specialization.

As an application of Theorem 3.1 we prove a relative limit theorem for functions defined by expressions of the form

$$u(s, \xi) = \int Q(s, \xi - \eta) U(d\eta). \tag{4.1}$$

Here ξ and η are points of Euclidean N-space, s is a point of a linear set with limit point 0, $Q(s, .)$ is a Baire function, and U is a signed measure of Borel subsets of N-space, finite for compact sets. It is supposed that $Q(s, \xi - .)$ is summable with respect to U for ξ in any compact set, if s is sufficiently small, depending on the set. The class of such signed measures U will be denoted by $\Gamma(Q)$, and includes all signed measures of bounded variation, since $Q(s, .)$ is a bounded function for each value of s under the hypotheses of the following theorem.

Theorem 4.1. Let U and H be in the class $\Gamma(Q)$. Suppose that

$$0 \leqq Q(s, \eta) \leqq M(s) = \sup_{\eta} Q(s, \eta) < \infty .$$

Let $\delta(.)$ be a strictly positive function on $(0, \infty)$, with limit 0 at 0. Suppose that the following conditions are satisfied.

(a) $\liminf_{s \to 0} \inf_{|\eta| \leqq \delta(s)} \dfrac{Q(s, \eta)}{M(s)} > 0 .$

(b) For every $\alpha > 0$,

$$Q_1(s, \alpha) = \sup_{|\eta| \geqq \alpha} Q(s, \eta) = o[\delta(s)^N] M(s) .$$

(c) There is a constant b for which

$$Q_1(s, |\eta|) \leqq bQ(s, \eta)$$

for all η satisfying $\delta(s) \leqq |\eta| \leqq \alpha$, if α is sufficiently small.

(d) If $\alpha > 0$, there is a number s_0, which can be taken arbitrarily near 0, for which

$$Q(s, \eta) \leqq o[\delta(s)^N] Q(s_0, \eta) M(s) \quad \text{when} \quad |\eta| \geqq \alpha .$$

Here $o[\delta(s)^N]$ may depend on α but not on η.

If u and h are defined as described above then, excluding a ζ-set of $\|H\|$ measure 0, independent of the choice of the function $\delta(.)$,

$$\lim_{\substack{s \to 0 \\ \xi \to \zeta}} \frac{u(s, \xi)}{h(s, \xi)} = \frac{dU}{dH}(\zeta) \tag{4.2}$$

if $\xi = \xi(s) \to \zeta$ in such a way that $|\xi(s) - \zeta| \leqq \delta(s)$ if $N = 1$ or $\limsup_{s \to 0} |\xi(s) - \zeta| / \delta(s) < 1$ if $N > 1$.

Before proving the theorem we remark that its application to integrals over a compact subset A of N-space is obtained by defining Q as 0 off the set, or, which amounts to the same thing, considering only signed measures vanishing on the Borel subsets of the complement of A. Under this condition U and H are necessarily of bounded variation on the domain of integration, so that condition (d) becomes irrelevant.

In the most important applications, $Q(s, \eta)$ defines a monotone decreasing function of $|\eta|$, for fixed s, so that $Q(s, \eta) = Q_1(s, |\eta|)$, and (c) is then satisfied with $b = 1$.

To prove Theorem 4.1, let ζ be a point of N-space at which the following conditions are satisfied: $(dU/dH)(\zeta)$ exists in the variational sense; $(dH/d\,\|H\|)(\zeta)$ exists in the variational sense and is 1 or -1; $\|H\|'(\zeta)$ $(\leqq \infty)$ exists and is strictly positive. These conditions exclude at most a set of $\|H\|$ measure 0, and we shall show that the excluded set will serve as the exceptional set of the theorem. Throughout the proof we shall suppose without further comment that s is so small that $u(s, \xi)$, $h(s, \xi)$ are defined in a neighborhood of ζ. Let $\{s_n, n \geqq 1\}$ be any sequence of points, with limit 0, in the domain of the first argument of Q, and let $\{\xi_n, n \geqq 1\}$ be any sequence of points of N-space satisfying the condition $|\xi_n - \zeta| \leqq \delta(s_n)$ if $N = 1$, or the stronger condition $\limsup_{n\to\infty} |\xi_n - \zeta| / \delta(s_n) < 1$ if $N > 1$. We now identify $K(n, \eta)$, η_n, δ_n, $U(d\eta)$, $H(d\eta)$ of Theorem 3.1 with

$$Q(s_n, \xi_n - \zeta - \eta)/M(s_n), \quad \xi_n - \zeta, \quad \delta(s_n), \quad U(\zeta + d\eta), \quad H(\zeta + d\eta)$$

respectively. Then the hypotheses of the present theorem correspond to those of Theorem 3.1, so that

$$\lim_{n\to\infty} \frac{u(s_n, \xi_n)}{h(s_n, \xi_n)} = \frac{dU}{dH}(\zeta), \tag{4,3}$$

as was to be proved.

Theorem 3.2 specializes to the following.

Theorem 4.2. *Theorem 4.1 remains valid if there is a hyperplane through the origin such that, for all s, $K(s, .)$ vanishes on the half-space strictly on one side of the hyperplane, if in conditions (a)–(d) η is restricted to be in the complementary closed half-space A, and if $\xi(s) \to \zeta$ in such a way that $\xi(s) - \zeta$ lies in A and, when $N > 1$, the distance of $\xi(s) - \zeta$ from the boundary of A is $\geqq \delta(s)/O(1)$.*

5. Applications.

To illustrate the applications of our relative limit theorems, we detail several special cases.

Case (A). Let Γ_A be the class of functions u on an open ball of radius R in Euclidean N-space, $N>1$, which are differences between pairs of positive harmonic functions there. Such a function has a Poisson-Stieltjes representation

$$u(\xi_0) = \int \frac{R^2 - |\xi_0|^2}{|\xi_0 - \eta|^N} U(d\eta), \tag{5.1}$$

where U is a signed measure of Borel subsets of the ball boundary, of bounded variation. To simplify notation we have assumed that the ball center is at the origin. If $s = R - |\xi_0|$, and if ξ is the point at which the ray from the center passing through ξ_0 meets the ball boundary, ξ_0 is determined by the pair (s, ξ), and u can be written in the form

$$u(\xi_0) = u(s, \xi) = \int Q(s, \xi - \eta) U(d\eta). \tag{5.2}$$

The integral here is an integral over the ball boundary which, locally at least, is an integral over $(N-1)$-dimensional sets. A trivial extension of Theorem 4.1, with N in that theorem replaced by $N-1$, is applicable to this case. We can choose $\delta(s) = \gamma s$, for any strictly positive constant γ. We thus obtain the following theorem.

Theorem 5.1. *Let u and h be in the class Γ_A, with Poisson-Stieltjes representations in terms of U, H respectively. Then, at $\|H\|$ almost every point ζ of the ball boundary, u/h has the limit $(dU/dH)(\zeta)$ on approach to ζ along non-tangential paths.*

This theorem was proved in [I] for u and h positive. We remark that, at the expense of losing the harmonic function interpretation, the boundary limit assertion remains true, with no change in proof, if the exponent N is replaced in (5.1) by any exponent $m > N-1$.

Case (B). In this example, the integration is over N-space, and U, H, need not be of bounded variation. Define Q by

$$Q(s, \xi) = s^{-N/2} e^{-|\xi|^2/2s}, \qquad 0<s<1, \tag{5.3}$$

and choose $\delta(s) = \gamma s^{1/2}$, for any strictly positive constant γ. Then Theorem 4.1 is applicable. Now if U is chosen so that $Q(s, \xi - .)$ is summable with respect to U for all pairs (ξ, s) in the strip S: $0<s<1$, then the

representation (4.1) is the Poisson-Stieltjes representation of a parabolic function ($u_s = \frac{1}{2}\Delta_\xi u$), valid if and only if the function is the difference between two positive parabolic functions on S. Let Γ_B be the class of such differences. Theorem 4.1 yields the following.

Theorem 5.2. *Let u and h be parabolic functions on S, in the class Γ_B, with Poisson-Stieltjes representations in terms of U and H respectively. Then, at $\|H\|$ almost every point $(0, \zeta)$ of the lower boundary of S, u/h has the limit $(dU/dH)(\zeta)$ on parabolic approach to $(0, \zeta)$.*

Here 'parabolic approach' means approach remaining in some paraboloid of revolution tangent to the lower boundary of S at $(0, \zeta)$ and opening out into S. This theorem is known [II] when $N=1$ and h is positive.

Case (C). Let $\{p_n, -\infty < n < \infty\}$ be a sequence of positive numbers, not all 0, and define

$$\Phi(\xi) = \sum_{-\infty}^{\infty} p_n \xi^n \quad , \quad \Psi(\xi) = -\xi\Phi'(\xi)/\Phi(\xi), \tag{5.4}$$

where it is supposed that the series for Φ converges in some open interval I. Define u by

$$u(n, \xi) = \int_I \frac{\eta^{n\xi}}{\Phi(\eta)^n} U(d\eta), \qquad n \geqq 0, \tag{5.5}$$

and define h similarly in terms of H. Here U and H are necessarily of bounded variation. It is easy to verify that the kernel in (5.5) is, for each ξ, a maximum as η varies when $\xi = \Psi(\eta)$, and is monotone on each side of the maximum. It has been shown [III] (at least for U and H positive, from which the general case can readily be deduced) that

$$\lim_{\substack{n\to\infty \\ \xi\to\zeta}} \frac{u(n, \xi)}{h(n, \xi)} = \frac{dU}{dH}(\zeta) \tag{5.6}$$

at $\|H\|$ almost every point ζ of I, if $\xi = \xi_n \to \Psi(\zeta)$ in such a way that

$$|\xi_n - \Psi(\zeta)| = O(n^{-1/2}). \tag{5.7}$$

(In the reference, $n\xi$ is integer-valued, but this is irrelevant to the proof.) This result cannot be deduced from Theorems 4.1 or 4.2. To apply Theorem 3.1 with $N=1$, define K by .

$$(5.8)\qquad K(n, \eta) = \frac{(\eta+\zeta)^{n\xi_n}\,\Phi(\eta_n+\zeta)^n}{\Phi(\eta+\zeta)^n(\eta_n+\zeta)^{n\xi_n}}, \qquad \xi_n = \Psi(\eta_n+\zeta),$$

where $\xi_n \to \Psi(\zeta)$. All the hypotheses of Theorem 3.1 except (d) are satisfied if δ_n of that theorem is accepted as $\gamma_n^{-1/2}$, for any strictly positive γ, large enough to match (5.7), and if η_n of that theorem is identified with η_n here, and, finally, if $U(d\eta)$, $H(d\eta)$ there are identified with $U(\zeta+d\eta)$, $H(\zeta+d\eta)$ here. Unfortunately K does not satisfy (d), but this fact causes no difficulty. In fact because of the monotoneity properties of the kernel, K satisfies a 'one-sided' version of (d), now to be described, under which Theorem 3.1 remains true. If K_1^+ $[K_1^-]$ is defined like K_1 in Theorem 3.1 (c) except that $\eta-\eta_n \geqq \alpha$ $[\eta-\eta_n \leqq -\alpha]$ in the definition, then the correspondingly modified version of (d) requires that

$$(5.9)\qquad \begin{aligned} K_1^+(n, \eta-r_n) &\leqq bK(n, \eta) \quad \text{if} \quad \delta_n \leqq \eta-\eta_n \leqq \alpha, \\ K_1^-(n, \eta-\eta_n) &\leqq bK(n, \eta) \quad \text{if} \quad \delta_n \leqq \eta_n-\eta \leqq \alpha. \end{aligned}$$

Now the only place (d) was used was in (3.12), and here, if K_1 has been replaced by K_1^+ and K_1^-, as made possible by carrying out the integration separately over the two intervals involved, the last term on the left in (3.12) is replaced by

$$(5.10)\qquad \varepsilon \int_{(\delta_n, \alpha]} K_1^+(n, r)\,H(dr)/h_n + \varepsilon \int_{[-\alpha, -\delta_n]} K_1^-(n, r)\,H(dr)/h_n.$$

Applying (5.9), we find that this sum is at most $b\varepsilon$, the same majorant obtained in (3.12). Thus Theorem 3.1 remains valid when $N=1$ under the one-sided version of (d). The corresponding one-sided modification of Theorem 4.1 is of course also true.

Case D. Let F be a monotone increasing function on $(-\infty, \infty)$, and define

$$(5.11)\qquad \Phi(\xi) = \int_{-\infty}^{\infty} e^{\xi\eta}\,dF(\eta), \qquad \Psi = \Phi'/\Phi,$$

where it is supposed that the integral converges on an open interval I. Define u by

$$u(n, \xi) = \int_I \frac{e^{n\xi\eta}}{\Phi(\eta)^n} U(d\eta), \tag{5.12}$$

and define h similarly in terms of H, where both signed measures are of bounded variation. If the support of F is a subset of the class of integers, Case (D) reduces to Case (C). In the present more general case an application of the one-sided version of Theorem 3.1 is still possible, and proves that (5.6) is true $\|H\|$ almost everywhere on I under the approach restriction (5.7). The fact that an analogue of Theorem 3.1 (see [II]) is applicable to obtain this result has already been pointed out [VII].

Case (E). We now give an example of the application of Theorem 4.2. Define u by

$$u(t, \xi) = t\int_\xi^\infty e^{-t(\eta-\xi)} U(d\eta) = t\int_0^\infty e^{-t\eta} U_\xi(d\eta), \quad U_\xi(d\eta) = U(\xi+d\eta), \tag{5.13}$$

where U is a signed measure of linear Borel sets, finite for compact sets. It is supposed that

$$\int_0^\infty e^{-t\eta} \|U\|(d\eta) < \infty \tag{5.14}$$

for sufficiently large t. Define h similarly in terms of H. Then we have the following theorem.

Theorem 5.3. *At $\|H\|$ almost every point ζ,*

$$\lim_{\substack{t\to\infty \\ \xi\to\zeta}} \frac{u(t, \xi)}{h(t, \xi)} = \frac{dU}{dH}(\zeta) \tag{5.15}$$

if $\xi = \xi(t) \to \zeta$ in such a way that $0 \leqq \zeta - \xi(t) = O(1/t)$.

To apply Theorem 4.2, identify s of that theorem with $1/t$ here, defining $Q(s, \xi)$ by

$$\begin{aligned} Q(s, \xi) &= e^{\xi/s}/s \quad \text{if } \xi \leq 0 \\ &= 0 \qquad\quad \text{if } \xi > 0, \end{aligned} \tag{5.16}$$

and set $\delta(s) = \gamma s$, where γ is any strictly positive constant. The theorem is well-known [IX, p. 182] when H is Lebesgue measure, so that $h = 1$.

Theorem 5.3 can be put into a probabilistic context. Define

$p(t_1, t_2, \xi, A)$, the probability of a transition from the point ξ at time $t_1 > 0$ to a point of the set A at time $t_2 > t_1$, by

$$p(t_1, t_2, \xi, \{\xi\}) = t_1/t_2$$
$$(5.17) \qquad p(t_1, t_2, \xi, d\eta) = \frac{t_1(t_2 - t_1)}{t_2} e^{-t_1(\eta-\xi)} d\eta, \qquad \text{if } \eta > \xi.$$

In words, ξ goes into itself with probability t_1/t_2; if not, it goes into $\eta > \xi$ in such a way that $\eta - \xi$ is distributed exponentially with expectation $1/t_1$. The transition probability distribution is that of a Markov process, that is, the Chapman-Kolmogorov equation is satisfied. If $\{x(t), t \geqq t_0 > 0\}$ is a Markov process with this transition probability function, any function u defined by (5.13) has the property that the $u[t, x(t)]$ process is a martingale if the initial random variable has a finite expectation, and, under appropriate side conditions, the converse is true. If the $x(t)$ process is separable, and if we neglect zero probabilities, the following statements are true. Each sample function of the $x(t)$ process is monotone increasing (wide sense) and has a finite limit when $t \to \infty$. The transition probability function under the condition that the sample function limit is ζ is given by

$$p^{\zeta}(t_1, t_2, \xi, \{\xi\}) = e^{-(t_2-t_1)(\zeta-\xi)}$$
$$(5.18) \qquad p^{\zeta}(t_1, t_2, \xi, d\eta) = (t_2 - t_1) e^{-(t_2-t_1)(\zeta-\eta)} d\eta, \quad \text{if } \xi < \eta < \zeta.$$

The ratio u/h has the limit $(dU/dH)(\zeta)$ on a probability path to the upper limit ζ. The latter fact can be deduced probabilistically, independently of Theorem 5.3.

Case (F). In all these examples, as an examination of the references for Cases (A)—(D) shows, and as the probability context in Case (E) shows, we have been dealing either with ratios of harmonic functions, as in Case (A), or with ratios of generalized harmonic functions. The averaging properties of such ratios are fundamental, and in fact these led to the probabilistic background which suggested the relative limit theorems. The following example does not seem to have a probabilistic context, however. Let $K^{(r)}$ be the kernel corresponding to (C, r) summability of a Fourier-Stieltjes transform of a signed measure U, so that the nth sum is

$$(5.19) \qquad u^{(r)}(n, \xi) = \int_{-\pi}^{\pi} K^{(r)}(n, \xi - \eta)\, U(d\eta).$$

Here

$$K^{(r)}(n, \xi) = \sum_{k=0}^{n} A_{n-k}^{(r-\alpha)} A_k^{(\alpha-1)} K^{(\alpha-1)}(k, \xi) / A_n^{(r)}, \tag{5.20}$$

where

$$K^{(0)}(n, \xi) = \frac{\sin(n+\frac{1}{2})\xi}{2\pi \sin \frac{1}{2}\xi} \tag{5.21}$$

and

$$A_n^{(r)} = (r+1)\dots(r+n)/n! \sim n^r/\Gamma(r+1), \quad r \neq -1, -2, \dots, \tag{5.22}$$

so that

$$A_n^{(r+s+1)} = \sum_{k=0}^{n} A_{n-k}^{(r)} A_k^{(s)}, \quad \sum_{k=0}^{\infty} A_k^{(r)} z^k = (1-z)^{-r-1}. \tag{5.23}$$

The function $K^{(r)}(n, .)$ is even, and

$$K^{(r)}(n, 0) = \frac{2n+1+r}{2\pi(r+1)} \sim \frac{n}{\pi(r+1)}. \tag{5.24}$$

If $r = 1$, $K^{(r)}$ is the Fejer kernel,

$$K^{(1)}(n, \xi) = \frac{1}{2\pi(n+1)} \left(\frac{\sin(n+1)\frac{\xi}{2}}{\sin\frac{\xi}{2}}\right)^2. \tag{5.25}$$

Applying (5.20) one can verify that certain properties of $K^{(r)}$ valid for specified values of r will be valid for larger values of r. The reference to (5.20) will be omitted below. For example, $K^{(r)}(n, .)$ has its maximum at 0 when $r = 0$ and therefore also when $r > 0$; $K^{(r)}$ is positive when $r = 1$ and therefore also when $r > 1$; $K^{(r)}(n, .)$ is monotone decreasing on $[0, \pi]$ when $r = 3$ [XII] and therefore also when $r > 3$.

We define $Q^{(r)}(s, .) = K^{(r)}(1/s, .)$ for s the reciprocal of a strictly positive integer and show that Theorem 4.1 is applicable to $Q^{(r)}$ when $r > 1$ and $\delta(s) = \gamma s$ for any strictly positive γ. To prove that condition (a) of Theorem 4.1 is satisfied we show that

$$K^{(r)}(n, \xi) \geqq a_1(\gamma, r)\, n \quad \text{if} \quad 0 \leqq \xi \leqq \gamma/n, \quad r > 1, \tag{5.26}$$

for a suitable strictly positive a_1. To see this we can suppose that $\gamma > \pi$ and then if $1 < r \leqq 2$ and if $n \geqq 2$,

$$K^{(r)}(n,\xi) = \sum_{k=0}^{n} A_{n-k}^{(r-2)} \frac{\sin^2 (k+1)\frac{\xi}{2}}{2\pi A_n^{(r)} \sin^2 \frac{\xi}{2}} \tag{5.27}$$

$$\geqq \frac{2A_n^{(r-2)}}{\pi^3 A_n^{(r)}} \sum_{k \leqq (n\pi/\gamma)-1} (k+1)^2 \geqq a_1(\gamma, r)\, n$$

$$\text{if } 0 \leqq \xi \leqq \gamma/n.$$

If $r>2$, (5.26) is valid, for properly chosen a_1, since it is valid for $r=2$.

To prove that condition (c) of Theorem 4.1 is satisfied, we show that

$$\sup_{\xi \geqq \eta} K^{(r)}(n,\xi) \leqq a_2(r)\, K^{(r)}(n,\eta) \quad \text{if} \quad r>1, \quad \eta \geqq 0, \tag{5.28}$$

for suitably chosen a_2. This inequality is trivial, with $a_2(r)=1$ when $r \geqq 3$ because of the monotoneity property of $K^{(r)}$ in that case, but we shall not use this fact. Apply (5.20) to obtain, after a trivial manipulation,

$$K^{(r)}(n,\xi) = \frac{A_n^{(r-1)} - \sum_{k=0}^{n} A_{n-k}^{(r-2)} \cos (k+1)\xi}{4\pi A_n^{(r)} \sin^2 \frac{\xi}{2}} \tag{5.29}$$

$$\leqq \frac{A_n^{(r-1)}}{2\pi A_n^{(r)} \sin^2 \frac{\xi}{2}} \qquad \text{if } r>1.$$

Note that the right hand side is $O(n^{-1}\xi^{-2})$, so that condition (b) of Theorem 4.1 for $Q^{(r)}$ is satisfied. If $1<r \leqq 2$, (5.23) can be applied, together with an Abel transformation, to yield the inequality

$$\left| \sum_{k=0}^{n} A_{n-k}^{(r-2)} \cos (k+1)\xi \right| \leqq \left| \sum_{j=0}^{n} A_j^{(r-2)} e^{-ij\xi} \right| \tag{5.30}$$

$$= \left| (1-e^{-i\xi})^{-r+1} - \sum_{j=n+1}^{\infty} A_j^{(r-2)} e^{-ij\xi} \right|$$

$$\leqq \left(2 \sin \frac{\xi}{2}\right)^{-r+1} + 2A_{n+1}^{(r-2)} \sin^{-1} \frac{\xi}{2}$$

$$\leqq A_n^{(r-1)}/2 \quad \text{if} \quad \xi \geqq a_3(r)/n$$

and if a_3 is sufficiently large. Combining (5.29) and (5.30) we obtain

$$\sup_{\xi \geqq \eta} K^{(r)}(n,\xi) \leqq 4K^{(r)}(n,\eta) \quad \text{if} \quad 1<r \leqq 2 \tag{5.31}$$

and if $\eta \geqq a_3(r)/n$. On the other hand, this restriction on η (always

assumed $\geqq 0$) is unnecessary in view of (5.24) and (5.26), if we are willing to increase the constant factor in (5.31). We have thus proved (5.28) for $1 < r \leqq 2$, and it must therefore be true for $r > 2$. (The function a_2 can be taken to be monotone non-increasing.)

Since we are dealing with a compact interval $[-\pi, \pi]$, our signed measures on the interval are necessarily of bounded variation, so that condition (d) of Theorem 4.1 is irrelevant. Our work has thus led to the following theorem.

Theorem 5.4. Let $u^{(r)}(n, .)$ and $h^{(r)}(n, .)$ be the Fourier-Stieltjes (C, r) nth sums corresponding to the signed measures U and H respectively. Then if $r > 1$, for $||H||$ almost every point ζ,

$$\lim_{\substack{n \to \infty \\ \xi \to \zeta}} \frac{u^{(r)}(n, \xi)}{h^{(r)}(n, \xi)} = \frac{dU}{dH}(\zeta) \tag{5.32}$$

whenever $\xi = \xi_n \to \zeta$ in such a way that $|\xi_n - \zeta| = O(1/n)$.

If H is Lebesgue measure and U is absolutely continuous, this theorem is classical. In fact Lebesgue [V] proved it in that case, even with $r = 1$ (but with $\xi = \zeta$ in (5.32)) and Hardy [IV] extended the validity to $r > 0$. W. H. Young [X] further extended the result to cover a not necessarily absolutely continuous U. See [XII] for a proof of Theorem 5.4 with $r > 0$, for H Lebesgue measure. The following example shows that the theorem cannot be extended to the case $r = 1$ without further restrictions. Let U be Lebesgue measure, and let H be the measure confined to the point 0, with $H(\{0\}) = 1$. Then if $r = 1$, $u^{(r)}$ is the constant function 1 and $h^{(1)} = K^{(1)}$. If $\xi_n = 2\pi/(n+1)$, $h^{(1)}(n, \xi_n) = 0$, whereas $u^{(1)}(n, \xi_n)/h^{(1)}(n, \xi_n)$ would have the limit 0, as $n \to \infty$ if the theorem were true for $r = 1$.

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VARIATION DIMINISHING HANKEL TRANSFORMS [1]

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1. Introduction.

If $\varphi(x)$ is a continuous function on a (possibly infinite) interval then we denote by $V[\varphi]$ the number of changes of sign of $\varphi(x)$ on the interval in question. A function $G(x) \in L^1(-\infty, \infty)$ is said to be a variation diminishing $*$-kernel if

$$V[G * \varphi] \leqq V[\varphi]$$

for every bounded continuous function $\varphi(x)$ defined on $(-\infty, \infty)$. Here

$$G * \varphi \cdot (x) = \int_{-\infty}^{\infty} G(x-y)\varphi(y)\,dy \qquad (-\infty < x < \infty).$$

In 1947 I. J. Schoenberg proved that G is a variation diminishing $*$-kernel if and only if (after multiplication by a suitable constant)

$$G^{\wedge}(t) = \int_{-\infty}^{\infty} G(x)\, e^{-itx}\,dx$$

$$= \left[e^{ct^2 + ibt} \prod_k \left(1 - \frac{it}{a_k}\right) e^{it/a_k}\right]^{-1},$$

where the a_k's are real and $\Sigma_k a_k^{-2}$ is finite, b is real, and c is real and non-negative.

In this paper we will establish an analogous theory associated with certain convolutions defined for functions on $(0, \infty)$. Let ν be fixed, [2] $\nu > 0$. We set

$$\mu(x) = x^{2\nu+1}\,[2^{\nu+1/2}\,\Gamma(\nu+\tfrac{3}{2})]^{-1} \tag{1}$$

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2. The case $\nu = 0$ is Schoenberg's theorem for an even kernel. While the arguments given here are valid for $\nu = 0$ many formulas require slight changes.

and we define $L^p(0, \infty)$, $1 \leqq p < \infty$, as the space of those real measurable functions on $(0, \infty)$ for which

$$\|f\|_p = \left[\int_0^\infty |f(x)|^p \, d\mu(x)\right]^{1/p}$$

is finite. Similarly $L^\infty(0, \infty)$ is the space of those real measurable functions on $(0, \infty)$ for which

$$\|f\|_\infty = \operatorname*{ess\,sup}_{0<x<\infty} |f(x)|$$

is finite. Let us further define

(2) $$D(x, y, z) = \frac{2^{(3\nu - \frac{5}{2})}\,\Gamma(\nu+\frac{1}{2})^2}{\Gamma(\nu)\,\pi^{1/2}} (xyz)^{-2\nu+1} A(x, x, z)^{2\nu-2}$$

where $A(x, y, z)$ is the area of a triangle whose sides are x, y, z if there is such a triangle and otherwise is zero. If $f(x)$ and $g(x)$ are real measurable functions on $(0, \infty)$ then we formally set

(3) $$f \# g \cdot (x) = \int_0^\infty \int_0^\infty f(y) g(z) D(x, y, z) \, d\mu(y) \, d\mu(z).$$

We will show in § 2 that the operation "#" has most of the familiar properties of the convolution operation "$*$" on the real line. In particular if $f(x) \in L^1(0, \infty)$ and $g(x) \in L^\infty(0, \infty)$ then the integral (3) is absolutely convergent for $0 < x < \infty$, $\|f \# g\|_\infty \leqq \|f\|_1 \|g\|_\infty$, and $f \# g \cdot (x)$ is continuous for $0 < x < \infty$. Similarly if $f(x) \in L^1(0, \infty)$ and $g(x) \in L^1(0, \infty)$ then the integral (3) is absolutely convergent for almost all x in $(0, \infty)$ and $\|f \# g\|_1 \leqq \|f\|_1 \|g\|_1$. Finally let

(4) $$\mathbf{J}(x) = 2^{\nu - 1/2}\,\Gamma(\nu + 1/2)\, x^{1/2 - \nu} J_{\nu - 1/2}(x).$$

We define

$$f^\wedge(t) = \int_0^\infty \mathbf{J}(xt) f(x) \, d\mu(x) \qquad 0 \leqq t < \infty;$$

$f^\wedge(t)$ is the Hankel transform of $f(x)$. The Hankel transform behaves in regard to the convolution "#" exactly as does the Fourier transform in

regard to ordinary convolution "$*$" on the real line; for example if $f, g \in L^1(0, \infty)$ then

$$(f \# g)^{\wedge}(t) = f^{\wedge}(t)\, g^{\wedge}(t).$$

Let $\psi(x)$ be a continuous function on $0 < x < \infty$ and let us denote by $V[\psi]$ the number of changes of sign of $\psi(x)$ on $0 < x < \infty$. A function $H(x) \in L^1(0, \infty)$ is said to be a variation diminishing $\#$-kernel if

$$V[H \# \psi] \leqq V[\psi]$$

for every bounded continuous ψ. The principal result of the present paper is that H is a variation diminishing $\#$-kernel if and only if (after multiplication by a suitable constant

$$H^{\wedge}(t) = \int_0^\infty H(x)\, \mathbf{J}(xt)\, d\mu(x)$$

$$= \left[e^{ct^2} \prod_k \left(1 + \frac{t^2}{a_k^2}\right)\right]^{-1},$$

where the a_k's are real and $\Sigma_k a_k^{-2}$ is finite, and c is non-negative.

Many important integral transforms, for example the Laplace transform and the Stieltjes transform, can be put in the form $f = G * \varphi$ where G is a variation diminishing $*$-kernel. Such integral transforms have an extensive theory, see [8]. The present paper makes it evident that almost all this theory can be transferred to the present context. See also in this connection [6].

2. Hankel transforms.

In this section we will develop systematically the theory of Hankel transforms. While most of this material is known it is nowhere systematically treated in the form we will require.

Using the notation developed in the introduction we have the following basic formula,

$$\int_0^\infty \mathbf{J}(zt)\, D(x, y, z)\, d\mu(z) = \mathbf{J}(xt)\, \mathbf{J}(yt) \tag{1}$$

valid for $0 < x < \infty$, $0 < y < \infty$, $0 \leqq t < \infty$ see [15; p. 367 and p. 411]. The special case $t = 0$ yields

$$(2) \qquad \int_0^{\infty} D(x, y, z)\, d\mu(z) = 1 .$$

We note that if $0 < x, y, z < \infty$ then

$$(3) \qquad D(x, y, z) \geqq 0 ;$$

we also note that

$$(4) \qquad D(x, y, z) = D(y, x, z) = D(x, z, y) \quad \text{etc.}$$

Theorem 2a. $|\mathbf{J}(x)| \leqq 1$ for $0 \leqq x < \infty$.

The function $\mathbf{J}(x)$ is continuous for $0 \leqq x < \infty$. The familiar asymptotic behavior of Bessel functions implies that $\lim_{x \to \infty} \mathbf{J}(x) = 0$. Let

$$m_1 = \underset{0 \leqq x < \infty}{\text{l.u.b.}} \mathbf{J}(x) .$$

If $m_1 > 1$ then since $\mathbf{J}(0) = 1$ there will be a value x_1, $0 < x_1 < \infty$ such that $\mathbf{J}(x_1) = m_1$. Setting $t = 1$, $x = x_1$, $y = x_1$ in (2) we find that

$$m_1^2 = \mathbf{J}(x_1)\, \mathbf{J}(x_1) = \int_0^{\infty} \mathbf{J}(z)\, D(x_1, x_1, z)\, d\mu(z)$$

$$\leqq \int_0^{\infty} m_1\, D(x_1, x_1, z)\, d\mu(z) = m_1 .$$

Here we have used the positivity of D and (2). This contradicts $m_1 > 1$. Similarly let

$$m_2 = \underset{0 \leqq x < \infty}{\text{g.l.b.}} \mathbf{J}(x) .$$

There is a value x_2, $0 < x_2 < \infty$, for which $\mathbf{J}(x_2) = m_2$. Let $t = 1$, $x = x_2$, $y = x_2$ in (1). We find that, using $m_1 = 1$, we have

$$m_2^2 = \mathbf{J}(x_2)\, \mathbf{J}(x_2) = \int_0^{\infty} \mathbf{J}(z)\, D(x_2, x_2, z)\, d\mu(z)$$

$$\leqq \int_0^{\infty} D(x_2, x_2, z)\, d\mu(z) = 1$$

and thus $m_2 \geqq -1$. A slightly more careful analysis shows that $|\mathbf{J}(x)| < 1$ if $x > 0$. The above proof is due to Bochner [2].

Let $L^p(0, \infty)$, $1 \leqq p \leqq \infty$ be defined as in § 1.

Theorem 2b. *Let* $1 \leqq r \leqq \infty$, $1 \leqq s \leqq \infty$ *and let* $1 \leqq p \leqq \infty$ *where*

$$\frac{1}{p} = \frac{1}{r} + \frac{1}{s} - 1 .$$

If $f \in L^r(0, \infty)$, $g \in L^s(0, \infty)$ *then the integral*

$$f \# g \cdot (x) = \int_0^\infty \int_0^\infty f(y)\, g(z)\, D(x, y, z)\, d\mu(y)\, d\mu(z) .$$

converges for almost all x, $0 < x < \infty$ *and*

$$\| f \# g \|_p \leqq \|f\|_r \cdot \|g\|_s . \tag{5}$$

Moreover if $p = \infty$ *then* $f \# g \cdot (x)$ *is defined for all* x $0 < x < \infty$ *and is continuous.*

That $f \# g \cdot (x)$ is defined almost everywhere and that (5) holds is a consequence of the relations

$$\int_0^\infty |D(x_1, x_2, x_3)|\, d\mu(x_i) = 1 \qquad i = 1, 2, 3$$

which follow from (2), (3) and (4). Since the argument is standard it is omitted. If $p = \infty$ then

$$\begin{aligned} f \# g \cdot (x) &- f \# g \cdot (x_0) \\ &= \int_0^\infty \int_0^\infty f(y)\, g(z)\, [D(x, y, z) - D(x_0, y, z)]\, d\mu(y)\, d\mu(z), \end{aligned}$$

$$| f \# g \cdot (x) = f \# g \cdot (x_0) | \leqq I_1 I_2 ,$$

where

$$I_1 = \left[\int_0^\infty \int_0^\infty |f(y)|^r\, |D(x, y, z) - D(x_0, y, z)|\, d\mu(y)\, d\mu(z) \right]^{1/r},$$

$$I_2 = \left[\int_0^\infty \int_0^\infty |g(z)|^s\, |D(x, y, z) - D(x_0, y, z)|\, d\mu(y)\, d\mu(z) \right]^{1/s}.$$

Since

$$\int_0^\infty |D(x, y, z) - D(x_0, y, z)|\, d\mu(z) \leqq 2$$

and since (as is easily verified)

$$\lim_{x\to x_0} \int_0^\infty |D(x, y, z) - D(x_0, y, z)|\, d\mu(z) = 0 \qquad 0<y<\infty$$

a simple application of the Lebesgue limit theorem implies that $\lim\limits_{x\to x_0} I_1 = 0$. Similarly $\lim\limits_{x\to x_0} I_2 = 0$, etc.

The property (4) implies that under the assumptions of Theorem 2b

(6) $$f \# g \cdot (x) = g \# f \cdot (x) \qquad \text{almost everywhere};$$

that is "$\#$" is commutative.

The next result concerns approximate identities.

Theorem 2c. If

1. $k_n(x) \geqq 0 \qquad (0<x<\infty)$,
2. $\int_0^\infty k_n(x)\, d\mu(x) = 1 \qquad (n = 0, 1, 2, \ldots)$,
3. $\lim\limits_{n\to\infty} \int_\delta^\infty k_n(x)\, d\mu(x) = 0 \qquad (\text{each } \delta>0)$,
4. $\varphi(x) \in L^\infty(0, \infty)$,
5. φ is continuous at x_0,

then

$$\lim_{n\to\infty} \varphi \# k_n \cdot (x_0) = \varphi(x_0).$$

We see using (2) that

$$I = \varphi \# k_n \cdot (x_0) - \varphi(x_0)$$

$$= \int_0^\infty \int_0^\infty [\varphi(x) - \varphi(x_0)]\, k_n(y)\, D(x_0, x, y)\, d\mu(x)\, d\mu(y).$$

Given $\varepsilon > 0$ let us choose $\delta > 0$ so small that

$$|\varphi(x) - \varphi(x_0)| < \varepsilon \quad \text{for} \quad |x - x_0| < \delta .$$

We now let

$$I_1 = \int_\delta^\infty \int_0^\infty [\varphi(x) - \varphi(x_0)]\, k_n(y)\, D(x_0, x, y)\, d\mu(x)\, d\mu(y),$$

$$I_2 = \int_0^\delta \int_0^\infty [\varphi(x) - \varphi(x_0)]\, k_n(y)\, D(x_0, x, y)\, d\mu(x)\, d\mu(y).$$

Now $|\varphi(x) - \varphi(x_0)| \leqq 2\, ||\varphi||_\infty$. Thus using (2)

$$|I_1| \leqq 2\, ||\varphi||_\infty \int_\delta^\infty \int_0^\infty k_n(y)\, D(x_0, x, y)\, d\mu(x)\, d\mu(y),$$

$$\leqq 2\, ||\varphi||_\infty \int_\delta^\infty k_n(y)\, d\mu(y),$$

and thus $\lim\limits_{n\to\infty} I_1 = 0$.

Referring back to the explicit form of D in § 1 we see that if $0 < y < \delta$ then $D(x_0, x, y)$ vanishes except when

$$x_0 - \delta \leqq x \leqq x_0 + \delta .$$

But in this range $|\varphi(x) - \varphi(x_0)| < \varepsilon$. Thus

$$I_2 \leqq \varepsilon \int_0^\delta \int_0^\infty k_n(y)\, D(x_0, x, y)\, d\mu(x)\, d\mu(y)$$

$$\leqq \varepsilon \int_0^\delta k_n(y)\, d\mu(y) \leqq \varepsilon .$$

It now follows that $\overline{\lim\limits_{n\to\infty}}\, |I| \leqq \varepsilon$, or since ε is arbitrary $\lim\limits_{n\to\infty} I = 0$, as desired.

Corollary 2c. Under the same assumptions on $k_n(x)$ if $f(x) \in L^1(0, \infty)$ then

(7) $$\lim_{n\to\infty} ||f \# k_n - f||_1 = 0 .$$

It is easily seen that if $g(x) \in L^1(0, \infty)$ and if $g_n(x) \in L^1(0, \infty)$ $n = 1, 2, \ldots$ then the conditions

1. $\lim\limits_{n\to\infty} g_n(x) = g(x)$ almost everywhere $0 < x < \infty$,
2. $\|g_n\|_1 \leq \|g\|_1$ $\quad n = 1, 2, \ldots,$

imply that

$$\lim_{n\to\infty} \|g - g_n\|_1 = 0 .$$

Using this principle (7) follows from Theorem 2c in the special case where $f(x) \in L^1(0, \infty)$ and in addition $f(x)$ is bounded and continuous. Since such functions are dense in $L^1(0, \infty)$ the general case follows by approximation.

For $f(x) \in L^1(0, \infty)$ we define, as in § 1,

$$f^\wedge(t) = \int_0^\infty f(x) \mathbf{J}(xt)\, d\mu(t) \qquad 0 \leqq t < \infty .$$

It is apparent from Theorem 2a that $f^\wedge(t)$ is continuous for $0 \leqq t < \infty$ and that

$$\|f^\wedge\|_\infty \leqq \|f\|_1 .$$

Theorem 2d. *If $f, g \in L^1(0, \infty)$ then*

$$(f \# g)^\wedge(t) = f^\wedge(t) g^\wedge(t) \qquad 0 \leqq t < \infty .$$

We have

$$f \# g \cdot (x) = \int_0^\infty \int_0^\infty f(y) g(z) D(x, y, z)\, d\mu(y)\, d\mu(z)$$

Multiplying by $\mathbf{J}(xt)$ and integrating with respect to $d\mu(x)$ we find that

$$(f \# g)^\wedge(t) = \int_0^\infty \mathbf{J}(xt)\, d\mu(x) \int_0^\infty \int_0^\infty f(y) g(z) D(x, y, z)\, d\mu(y)\, d\mu(z)$$

$$= \int_0^\infty \int_0^\infty f(y) g(z)\, d\mu(y)\, d\mu(z) \int_0^\infty D(x, y, z) \mathbf{J}(xt)\, d\mu(x) .$$

The change of order in the integrations is easily justified using Fubini's theorem and (2).

Making use of (1) we see that

$$(f \# g)^{\wedge}(t) = \int_0^\infty \int_0^\infty f(y)\, g(t)\, \mathbf{J}(yt)\, \mathbf{J}(zt)\, d\mu(y)\, d\mu(z)$$

$$= f^{\wedge}(t)\, g^{\wedge}(t),$$

as desired.

Let us choose a sequence of functions $k_n(x)$ $n = 0, 1, 2, \ldots$ such that:

1. $k_n(x) \geqq 0$ $\qquad 0 < x < \infty;$
2. $\int_0^\infty k_n(x)\, dx = 1;$
3. $\lim_{n \to \infty} \int_\varepsilon^\infty k_n(x)\, dx = 0$ $\qquad$ for each $\varepsilon > 0;$
4. $k_n^{\wedge}(t) \in L^1(0, \infty);$
5. $k_n^{\wedge\wedge}(x) = k_n(x).$

One such sequence of functions will be exhibited at the end of the proof of the following result.

Theorem 2c. *If $k_n(x)$ $n = 0, 1, \ldots$ has the properties 1—5 above then for any $f \in L^1(0, \infty)$ we have*

$$\lim_{n \to \infty} \| f(x) - (f^{\wedge} k_n^{\wedge})^{\wedge}(x) \|_1 = 0.$$

By definition

$$(f^{\wedge} k_n^{\wedge})^{\wedge}(x) = \int_0^\infty k_n^{\wedge}(t)\, f^{\wedge}(t)\, \mathbf{J}(xt)\, d\mu(t)$$

$$= \int_0^\infty k_n^{\wedge}(t)\, \mathbf{J}(xt)\, d\mu(t) \int_0^\infty f(y)\, \mathbf{J}(yt)\, d\mu(y)$$

$$= \int_0^\infty f(y)\, d\mu(y) \int_0^\infty k_n^{\wedge}(t)\, \mathbf{J}(xt)\, \mathbf{J}(yt)\, d\mu(y)$$

$$= \int_0^\infty f(y)\, d\mu(y) \int_0^\infty k_n^\wedge(t)\, d\mu(t) \int_0^\infty \mathbf{J}(zt)\, D(x, y, z)\, d\mu(z)$$

$$= \int_0^\infty f(y)\, d\mu(y) \int_0^\infty D(x, y, z)\, d\mu(z) \int_0^\infty k_n^\wedge(t)\, \mathbf{J}(zt)\, d\mu(t)$$

$$= \int_0^\infty f(y)\, d\mu(y) \int_0^\infty D(x, y, z)\, k_n(z)\, d\mu(z)$$

$$= \int_0^\infty \int_0^\infty f(y)\, k_n(z)\, D(x, y, z)\, d\mu(y)\, d\mu(z)$$

$$= f \# k_n \cdot (x).$$

Here all the various changes of orders of integrations are justified by easy applications of Fubini's theorem. Our demonstration may now be completed by appealing to Corollary 2c.

An example of a set of functions satisfying the conditions $1-5$ above is

$$k_n(x) = e^{-nx}\, n^{2\nu+1}\, 2^{\nu-1/2}\, \Gamma(\nu + 1/2) / \Gamma(2\nu + 1).$$

Clearly Conditions 1, 2 and 3 are met. Since

$$k_n^\wedge(t) = \left[1 + \frac{t^2}{n^2}\right]^{-\nu-1},$$

see Watson [15; p. 386], Condition 4 is satisfied. Finally Condition 5 is a consequence of formula (2) of Watson [15; p. 434].

Corollary 2e. *If $f(x) \in L^1(0, \infty)$ and if $f^\wedge(t) \in L^1(0, \infty)$ then $f(x)$ may be redefined on a set of measure 0 so that it is continuous $0 < x < \infty$, and then*

$$f(x) = \int_0^\infty f^\wedge(t)\, \mathbf{J}(x)\, d\mu(t) \qquad 0 < x < \infty.$$

An immediate consequence of Theorem 2c is that if $f, g, h \in L^1(0, \infty)$ then

$$(f \# g) \# h \cdot (x) = f \# (g \# h) \cdot (x) \tag{8}$$

almost everywhere for $0 < x < \infty$; that is "$\#$" is associative. Using Theorem 2d we see that

$$[(f \# g) \# h]^{\wedge} = (f^{\wedge} g^{\wedge}) h^{\wedge} = f^{\wedge} (g^{\wedge} h^{\wedge}) = [f \# (g \# h)]^{\wedge}$$

from which using Theorem 2c we deduce that

$$\| f \# (g \# h) - (f \# g) \# h \|_1 = 0 .$$

This implies (8).

For future reference we remark that (8) is also valid for all x, $0 < x < \infty$, under the assumptions $f, g \in L^1(0, \infty)$, $h \in L^\infty(0, \infty)$. This can be deduced by an evident approximation argument.

We conclude this section by verifying certain differential relations. Let us set

$$\triangle h(x) = h''(x) + \frac{2\nu}{x} h'(x) .$$

We assert that

(9) $$\triangle \mathbf{J}(xt) = -t^2 \mathbf{J}(xt) .$$

To see this let $z = xt$. Then if $c_\nu = 2^{\nu-1/2} \Gamma(\nu + 1/2)$

$$\mathbf{J}(xt) = c_\nu z^{1/2-\nu} J_{\nu-1/2}(z) ,$$

$$\frac{d}{dx} \mathbf{J}(xt) = c_\nu t \frac{d}{dz} [z^{1/2-\nu} J_{\nu-1/2}(z)] ,$$

$$\left(\frac{d}{dx}\right)^2 \mathbf{J}(xt) = c_\nu t^2 \left(\frac{d}{dz}\right)^2 [z^{1/2-\nu} J_{\nu-1/2}(z)] .$$

Thus

$$\begin{aligned} \triangle \mathbf{J}(xt) &= c_\nu t^2 \left(\frac{d}{dz}\right)^2 [z^{1/2-\nu} J_{\nu-1/2}(z)] \\ &\quad + c_\nu \frac{2\nu t^2}{z} \frac{d}{dz} [z^{1/2-\nu} J_{\nu-1/2}(z)] \\ &= c t_\nu^2 z^{1/2-\nu} \left[J''_{\nu-1/2} + \frac{1}{z} J'_{\nu-1/2}(z) - \frac{1}{z^2} (\nu - 1/2)^2 J_{\nu-1/2}(z) \right] \\ &= -c_\nu t^2 z^{1/2-\nu} [J_{\nu-1/2}(z)] = -t^2 \mathbf{J}(xt) . \end{aligned}$$

In the next to last step we have used the relation

$$z^2 J''_{\nu-1/2}(z) + z J'_{\nu-1/2}(z) + [z^2 - (\nu - 1/2)^2] J_{\nu-1/2}(z) = 0,$$

see Watson [15; p. 82].

Let us also set

$$\mathbf{I}(x) = [2^{\nu-1/2}\Gamma(\nu + 1/2)]^{-1} I_{\nu-1/2}(x)\, x^{1/2-\nu}, \tag{10}$$

$$\mathbf{K}(x) = K_{\nu-1/2}(x)\, x^{1/2-\nu}. \tag{11}$$

We assert that

$$\begin{aligned} \triangle\, \mathbf{I}(xt) &= t^2\, \mathbf{I}(xt), \\ \triangle\, \mathbf{K}(xt) &= t^2\, \mathbf{K}(xt). \end{aligned} \tag{12}$$

This is proved exactly as (9) using

$$z^2 I''_{\nu-1/2}(z) + z I'_{\nu-1/2}(z) - [z^2 + (\nu - 1/2)^2] I_{\nu-1/2}(z) = 0,$$

etc. Finally a routine result about Wronskians shows that

$$\mathbf{K}(xt)\frac{d}{dx}\mathbf{I}(xt) - \mathbf{I}(xt)\frac{d}{dx}\mathbf{K}(xt) = a_\nu x^{-2\nu} \tag{13}$$

where a_ν is independent of x.

3. Elementary kernels.

A real function $f(x)$ defined for $0 < x < \infty$ is said to have at least n changes of sign if there exist numbers

$$0 < t_0 < t_1 < \ldots < t_n$$

such that

$$f(t_i)\, f(t_{i-1}) < 0 \qquad i = 1, \ldots, n.$$

f has exactly n changes of sign if it has at least n changes of sign and does not have at least $n + 1$ changes of sign. The number of changes of sign of f is denoted by $V[f]$; $V[f]$ has one of the values 0, 1, 2, ... or $+\infty$.

We denote by C the subset of $L^\infty(0, \infty)$ consisting of functions continuous for $0 < x < \infty$.

Definition 3a. *A function $K(x) \in L^1(0, \infty)$ is said to be a variation diminishing kernel if for every $f \in C$ we have*

$$V[K \# f] \leq V[f].$$

In the present section we will exhibit a family of simple variation-diminishing kernels.

Let S be the subclass of C consisting of functions $f(x)$ infinitely differentiable and vanishing outside a compact subset of $0 < x < \infty$.

Lemma 3b. *If $f(x) \in S$ then $f^{\wedge}(t) \in L^1(0, \infty)$.*

Proof. We have

$$f^{\wedge}(t) = \int_0^\infty f(x) \mathbf{J}(xt) \, d\mu(x).$$

If we first use the identity

$$\frac{-1}{t^2} \left[\frac{d^2}{dx^2} + \frac{2\nu}{x} \frac{d}{dx} \right] \mathbf{J}(xt) = \mathbf{J}(xt)$$

and then integrate by parts we find that

$$t^2 f^{\wedge}(t) = \int_0^\infty [Lf(x)] \mathbf{J}(xt) \, d\mu(x)$$

where

$$Lf(x) = -x^{-2\nu} \left(\frac{d}{dx} \right)^2 x^{2\nu} f(x) + 2\nu x^{-2\nu} \frac{d}{dx} x^{2\nu-1} f(x).$$

Clearly $Lf \in S$. Repeated application of this identity gives

$$t^{2n} f^{\wedge}(t) = \int_0^\infty [L^n f(x)] \mathbf{J}(xt) \, d\mu(x)$$

from which our assertion easily follows.

We require the following elementary results. Let $A(x, t)$ and $\frac{\partial}{\partial x} A(x, t)$ be continuous functions of x and t for $0 < t, \; x < \infty$, and let

$$\begin{aligned} A(x, t) &\leq c_1(t) \\ \left| \frac{\partial}{\partial x} A(x, t) \right| &\leq c_2(t) \end{aligned} \qquad 0 < t, \; x < \infty.$$

We assert that if

$$\int_0^\infty |\varphi(t)|\, c_1(t)\, dt < \infty$$

then

$$\psi(x) = \int_0^\infty \varphi(t)\, A(x, t)\, dt$$

is well defined and continuous for $0 < x < \infty$, and that if in addition

$$\int_0^\infty |\varphi(t)|\, c_2(t)\, dt$$

then

$$\frac{d}{dx}\psi(x) = \int_0^\infty \varphi(t) \frac{\partial}{\partial x} A(x, t)\, dt \qquad 0 < x < \infty,$$

and further $\frac{d}{dx}\psi(x)$ is continuous. The proofs of these assertions are familiar and are omitted.

For $a > 0$ let us set

$$g_a(x) = a^{2\nu+1}\mathbf{K}(ax)$$

where $\mathbf{K}$ is defined as in (10) of § 2. The object of the present section is to prove that $g_a(x)$ is a variation diminishing kernel.

Lemma 3c. $g_a\hat{}\,(t) = \left[1 + \frac{t^2}{a^2}\right]^{-1} \qquad 0 \leqq t < \infty.$

This follows from a formula of Watson [15; p. 410].

Lemma 3d. If $f \in L^1(0, \infty)$ and is such that $f\hat{} \in L^1(0, \infty)$ and if

$$h(x) = g_a(x) \# f(x)$$

then

$$\left(1 - \frac{\Delta}{a^2}\right) h(x) = f(x)$$

where

$$\triangle h(x) = h''(x) + \frac{2\nu}{x} h'(x).$$

By Theorem 2d we have

$$h^{\wedge}(t) = f^{\wedge}(t)\left(1 + \frac{t^2}{a^2}\right)^{-1}.$$

Since $h^{\wedge}(t) \in L^1(0, \infty)$ it follows by Corollary 2e that

$$h(x) = \int_0^\infty f^{\wedge}(t)\left[1 + \frac{t^2}{a^2}\right]^{-1} \mathbf{J}(xt)\, d\mu(t) \qquad 0 < x < \infty.$$

Now from the asymptotic formulas for Bessel functions it is easily seen that there exists a constant A such that

$$\left.\begin{aligned} \left|\frac{\partial}{\partial x}\mathbf{J}(xt)\right| &\leqq At \\ \left|\frac{\partial^2}{\partial x^2}\mathbf{J}(xt)\right| &\leqq At^2 \end{aligned}\right. \qquad 0 < t,\ x < \infty.$$

The discussion preceding Lemma 3c now shows that with the assumptions we have made

$$\left[1 - \frac{\triangle}{a^2}\right] h(x) = \int_0^\infty f^{\wedge}(t)\left[1 + \frac{t^2}{a^2}\right]^{-1}\left[1 + \frac{\triangle}{a^2}\right]\mathbf{J}(xt)\, d\mu(t).$$

Since

$$\left[1 - \frac{\triangle}{a^2}\right]\mathbf{J}(xt) = \left[1 + \frac{t^2}{a^2}\right]\mathbf{J}(xt),$$

see (9) § 2, we have

$$\left[1 - \frac{\triangle}{a^2}\right] h(x) = \int_0^\infty f^{\wedge}(t)\,\mathbf{J}(xt)\, d\mu(t) = f(x),$$

where the last equality is justified by a second application of Corollary 2e.

Lemma 3e. *If* $\mathbf{I}(x)$ *is defined as in* § 2 *then*

$$(1) \qquad (\Delta - a^2)\, h(x) = \frac{1}{x^{2\nu}\mathbf{I}(ax)}\left(\frac{d}{dx}\right)\left\{x^{2\nu}\mathbf{I}(ax)^2\left(\frac{d}{dx}\right)[h(x)/\mathbf{I}(ax)]\right\},$$

$$(2) \qquad = \frac{1}{x^{2\nu}\mathbf{K}(ax)}\left(\frac{d}{dx}\right)\left\{x^{2\nu}\mathbf{K}(ax)^2\left(\frac{d}{dx}\right)[h(x)/\mathbf{K}(ax)]\right\}.$$

The differential operators on both sides of these relations are of the form

$$h''(x) + A(x)\,h'(x) + B(x)\,h(x).$$

If we can find two linearly independent functions such that both functions are annihilated by each of the operators in question then the operators must coincide. Now it follows from (10) of § 2 that $\mathbf{I}(ax)$ and $\mathbf{K}(ax)$ are annihilated by $\left(1 - \frac{\Delta}{a^2}\right)$. Consider the operator on the right of (1). It clearly annihilates $h(x) = \mathbf{I}(ax)$. Using (11) of § 2 it is apparent that it annihilates $\mathbf{K}(ax)$ as well. This proves (1) and (2) is entirely similar.

Theorem 3f. *For $a > 0$ $g_a(x)$ is a variation diminishing kernel.*

Proof. Suppose first that $f(x) \in S$. Then by Lemma 3d if $h = g_a \# f$ we have

$$f(x) = \left(1 - \frac{\Delta}{a^2}\right) h(x).$$

We require the following simple remark. If $h(x)$ is a continuously differentiable function for $0 < x < \infty$, and if $\Omega(x)$ is a positive continuously differentiable function such that either of the conditions

$$\lim_{x \to 0+} \Omega(x)\,h(x) = 0 \quad \text{or} \quad \lim_{x \to \infty} \Omega(x)\,h(x) = 0$$

holds then

$$V\left[\frac{d}{dx}\,\Omega(x)\,h(x)\right] \geq V[h(x)].$$

If $f \in S$ then $h^{(n)}(x) \in C$ for $n = 0, 1, 2, \ldots$. Using this, the relations

$$\lim_{x \to \infty} 1/\mathbf{I}(ax) = 0, \quad \lim_{x \to 0+} x^{2\nu}\mathbf{I}(ax)^2 = 0,$$

the remark above, and formula (1) of Lemma 3e, we see that $V[f] \geq V[h]$.

If f is continuous and bounded then there exist a sequence of functions $f_n(x)$ on $0<x<\infty$ such that

1. $\|f_n\|_\infty \leqq \|f\|_\infty$,
2. $\lim_{n\to\infty} f_n(x) = f(x) \qquad 0<x<\infty$,
3. $f_n \in S$,
4. $V[f_n] \leq V[f]$.

Let $h_n = g_a \# f_n$. By the Lebesgue limit theorem $\lim_{n\to\infty} h_n(x) = h(x)$, $0<x<\infty$. For each n we have $V[h_n] \leqq V[f_n] \leqq V[f]$ and thus passing to the limit $V[h] \leqq V[f]$, q.e.d.

4. Composite kernels.

In this section we will use the simple variation diminishing kernels found in the preceding section in order to build up more complex variation diminishing kernels.

Lemma 4a. *If $K_1(x)$ and $K_2(x)$ are variation diminishing kernels then $K = K_1 \# K_2$ is also a variation diminishing kernel.*

Proof. Let $f \in C$. Then

$$V[K \# f] = V[K_1 \# (K_2 \# f)] \leqq V[K_2 \# f] \leqq V[f]$$

where we have utilized in succession the variation diminishing properties of K_1 and K_2.

Theorem 4b. *Let $c \geqq 0$ and*[3] *$0 < a_1 \leqq a_2 \leqq \ldots$ where $\Sigma_k a_k^{-2} < \infty$. If*

$$E(t) = e^{ct^2} \prod_k \left[1 + \frac{t^2}{a_k^2}\right]$$

then $1/E(t)$ is the Hankel transform of a variation diminishing kernel $G(x)$,

$$\int_0^\infty \mathbf{J}(xt)\, G(x)\, d\mu(x) = 1/E(t),$$

3. There may be no a_k's or finitely many a_k's or infinitely many a_k's. However we do exclude the case $c=0$ and no a_k's; i.e. $E(t) \equiv 1$.

Proof. By Lemma 4a and Theorem 3f if $E(t)$ is of the form

$$\prod_1^n \left[1+\frac{t^2}{a_k^2}\right]$$

then our theorem is true with

$$G(x) = g_{a_1}(x) \# \ldots \# g_{a_n}(x).$$

Consider next the case

$$E(t) = \prod_1^\infty \left[1+\frac{t^2}{a_k^2}\right]. \tag{1}$$

Let us set

$$G(x) = \int_0^\infty [E(t)]^{-1} \mathbf{J}(xt)\, d\mu(t).$$

Since $E(t) \in L^1(0, \infty)$, $G(x) \in C$. Let

$$E_n(t) = \prod_1^n \left[1+\frac{t^2}{a_k^2}\right].$$

Then if $G_n = g_{a_1} \# \ldots \# g_{a_n}$ we have

$$\int_0^\infty \mathbf{J}(xt)\, G_n(x)\, d\mu(t) = 1/E_n(t).$$

If n is sufficiently large $(2n > 2\nu + 1)$ then $[E_n(t)]^{-1} \in L^1(0, \infty)$ and therefore

$$G_n(x) = \int_0^\infty [E_n(t)]^{-1} \mathbf{J}(xt)\, d\mu(t),$$

see Corollary 2e. Thus

$$G(x) - G_n(x) = \int_0^\infty \{[E(t)]^{-1} - [E_n(t)]^{-1}\} \mathbf{J}(xt)\, d\mu(t)$$

and from this it is easy to show that

(2) $$\lim_{n\to\infty} G_n(x) = G(x),$$

uniformly for $0<x<\infty$. Since each $G_n(x) \geqq 0$ it follows that

(3) $$G(x) \geqq 0 \qquad 0<x<\infty\,.$$

Finally since

(4) $$\int_0^\infty G_n(x)\,d\mu(x) = 1 \qquad n = 0, 1, \ldots$$

it follows from Fatou's Lemma that

$$\int_0^\infty G(x)\,d\mu(x) \leq 1\,.$$

Thus $G \in L^1(0, \infty)$. Appealing to Corollary 2e we now see that

(5) $$\int_0^\infty \mathbf{J}(xt)\,G(x)\,d\mu(x) = 1/E(t)\,.$$

We have thus shown that $1/E(t)$ is indeed the Hankel transform of a function $G(x) \in L^1(0, \infty)$. It remains to prove that $G(x)$ is variation diminishing. Letting $t \to 0+$ in (5) we find that

(6) $$\int_0^\infty G(x)\,d\mu(x) = 1\,.$$

The above information in conjunction with the remarks following Corollary 2c implies that

$$\lim_{n\to\infty} \| G(x) - G_n(x) \|_1 = 0\,.$$

Thus if $f(x) \in C$ then

$$G \# f \cdot (x) = \lim_{n\to\infty} G_n \# f \cdot (x) \qquad 0<x<\infty,$$

and thus

$$V[G \# f] \leq \varliminf_{n\to\infty} V[G_n \# f] \leqq V[f]\,.$$

The case

$$(1) \qquad E(t) = e^{ct^2} \qquad (c > 0)$$

is dealt with in almost the same fashion. The only difference is that we put

$$E_n(t) = \left[1 + \frac{ct^2}{n}\right]^n .$$

Finally the general case can be obtained by combining the cases we have treated.

We note for future reference that if

$$(8) \qquad h_c^{\wedge}(t) = e^{-ct^2}$$

then

$$(9) \qquad h_c(x) = (2c)^{-\nu-1/2} e^{-x^2/4c},$$

see Watson [15; p. 394].

5. Order properties.

In this section we will essentially show that every variation diminishing kernel vanishes exponentially at $+\infty$. We begin with some elementary remarks on normalization.

Definition 5a. *A measurable function $h(x)$ on $0 < x < \infty$ is said to be a (probability) density function if*

$$0 \leqq h(x) \qquad 0 < x < \infty ,$$

$$\int_0^\infty h(x)\, d\mu(x) = 1 .$$

Clearly any finite convolution of density functions is again a density function.

Lemma 5b. *If $h(x)$ is a variation diminishing kernel then either $h(x) \geqq 0$ or $h(x) \leqq 0$ almost everywhere on $0 < x < \infty$.*

Proof. By Corollary 2c

$$\lim_{n \to \infty} \| h - h \# k_n \|_1 = 0 .$$

Here $k_n(x)$ is any sequence of functions satisfying conditions 1—3 of

Theorem 2c. If the conclusion of our lemma is false then for every sufficiently large n we will have $V[h \# k_n] \geqq 1$. But since h is variation diminishing we must have $V[h \# k_n] \leqq V[k_n] = 0$. This contradiction proves our result.

It now follows that if $h(x)$ is a variation diminishing kernel then $a^{-1} h(x)$ will be a variation diminishing density function if

$$a = \int_0^\infty h(x)\, d\mu(x).$$

Let $G(x)$ be a variation diminishing density function. We set

$$G_1 = G \# h_1.$$

Here h_c is defined by (9) of § 4. The advantage of replacing G by G_1 is that G_1, which is also a variation diminishing frequency function, is infinitely differentiable for $0 < x < \infty$.

Lemma 5c. Let $a > 0$. Then $G_1(x)/\mathbf{K}(ax)$ does not have a local minimum in $0 < x < \infty$.

We first note that the function

$$\left[1 - \frac{\Delta}{a^2}\right] h_c(x) = h_c(x)\left[1 + \frac{1+2\nu}{2c\,a^2}\, x - \frac{1}{4c^2\,a^2}\, x^2\right]$$

has one change of sign in $0 < x < \infty$. It follows that since G_1 is a variation diminishing kernel

$$G_1(x) \# \left[1 - \frac{\Delta}{a^2}\right] h_c(x) \tag{1}$$

has at most one change of sign for $0 < x < \infty$.

Now the function

$$\left[1 - \frac{\Delta}{a^2}\right] G_1(x) \# h_c(x) \tag{2}$$

has the same "ˆ" as the function above. Thus since both functions are in $L^1(0, \infty)$ they are equal almost everywhere and since they are continuous they are equal everywhere for $0 < x < \infty$. Letting $c \to 0+$ we find, see Theorem 2c, that

$$\left[1 - \frac{\Delta}{a^2}\right] G_1(x)$$

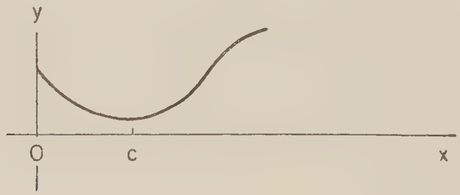
has at most one change of sign in $0 < x < \infty$.

Now by Lemma 3e

$$(\Delta - a^2)\, G_1(x) = \frac{-1}{x^{2\nu}\mathbf{K}(ax)} \left(\frac{d}{dx}\right) \left\{ x^{2\nu}\mathbf{K}(ax)^2 \left(\frac{d}{dx}\right) \frac{G_1(x)}{\mathbf{K}(ax)} \right\}.$$

We must distinguish two cases: i. $0 < \nu < 1/2$ and ii. $1/2 \leqq \nu < \infty$.

Case i.



$y = G_1(x)/\mathbf{K}(ax)$

If $G_1(x)/\mathbf{K}(ax)$ had a local minimum at c then $\dfrac{d}{dx}\,[G_1(x)/\mathbf{K}(ax)]$ would have some negative values to the left of c and some positive values to the right of c. Now it is easy to see that

$$\lim_{x\to\infty} x^{2\nu}\mathbf{K}(ax)^2 \left(\frac{d}{dx}\right) \frac{G_1(x)}{\mathbf{K}(ax)} = 0, \tag{3}$$

$$\lim_{x\to 0+} x^{2\nu}\mathbf{K}(ax)^2 \left(\frac{d}{dx}\right) \frac{G_1(x)}{\mathbf{K}(ax)} \geqq 0. \tag{4}$$

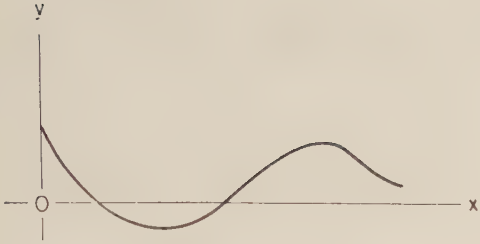
To see that (3) holds we note that $G_1(x)$, $G'_1(x)$ are $O(1)$ as $x \to \infty$ while $\mathbf{K}(ax)$, $\mathbf{K}'(ax)$ vanish exponentially as $x \to \infty$. To see that (4) holds we note that

$$x^{2\nu}\mathbf{K}(ax)^2 \left(\frac{d}{dx}\right) \frac{G_1(x)}{\mathbf{K}(ax)} = -x^{2\nu} G_1(x) \frac{d}{dx}\mathbf{K}(ax) + x^{2\nu}\mathbf{K}(ax) \frac{d}{dx} G_1(x)$$

from which it follows that

$$\begin{aligned} \lim_{x\to 0+} x^{2\nu}\mathbf{K}(ax)^2 \left(\frac{d}{dx}\right) \frac{G_1(x)}{\mathbf{K}(ax)} &= -G_1(0+) \lim_{x\to 0} x^{2\nu} \frac{d}{dx}\mathbf{K}(ax) \\ &= d_\nu G_1(0+) \end{aligned}$$

where $d_\nu > 0$.



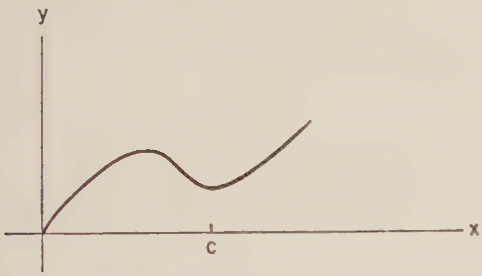
$$y = x^{2\nu}\,\mathbf{K}(ax)^2\left(\frac{d}{dx}\right)\frac{G_1(x)}{\mathbf{K}(ax)}$$

Clearly $x^{2\nu}\,\mathbf{K}(ax)^2\,\frac{d}{dx}\left(\frac{G_1(x)}{\mathbf{K}(ax)}\right)$ must be successively decreasing, increasing, and decreasing and thus its derivative must have at least two changes of sign, a contradiction.

Case ii. $\nu \geqq 1/2$. Here $\mathbf{K}(ax)$ approaches infinity as $x \to 0+$ and thus

(5) $$\lim_{x\to 0+} G_1(x)/\mathbf{K}(ax) = 0\,.$$

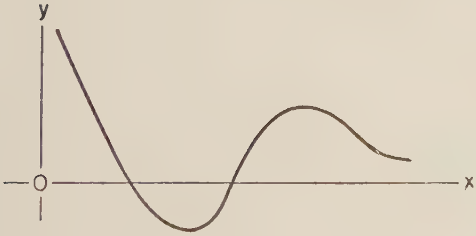
If $G_1(x)/\mathbf{K}(ax)$ has a local minimum at $x = c$



$$y = G_1(x)/\mathbf{K}(ax)$$

then $\frac{d}{dx}\,G_1(x)/\mathbf{K}(ax)$ must be successively positive, negative, and positive. As before

(6) $$\lim_{x\to\infty} x^{2\nu}\,\mathbf{K}(ax)^2\left(\frac{d}{dx}\right)\frac{G_1(x)}{\mathbf{K}(ax)} = 0$$



$$y = x^{2\nu}\,\mathbf{K}(ax)^2\left(\frac{d}{dx}\right)\frac{G_1(x)}{\mathbf{K}(ax)}$$

Thus the derivative of $x^{2\nu}\mathbf{K}(ax)^2\left(\frac{d}{dx}\right)\frac{G_1(x)}{\mathbf{K}(ax)}$ must have two (or more) changes of sign, a contradiction.

Theorem 5d. *There is a value $a>0$ such that*

$$G_1(x)=O[\mathbf{K}(ax)] \qquad x\to+\infty.$$

Proof. We know from Lemma 5c that for a given $a>0$, $G_1(x)/\mathbf{K}(ax)$ is either non-decreasing for $0<x<\infty$ or it is eventually non-increasing. In the latter case we have our desired conclusion. Thus what we must do is to rule out $G_1(x)/\mathbf{K}(ax)$ non-increasing on $0<x<\infty$ for every $a>0$. There are three cases: i. $0<\nu<1/2$; ii. $\nu=1/2$; and iii. $\nu>1/2$.

Case i. Here $\lim\limits_{x\to0+}\mathbf{K}(x)=c>0$ (c depends on ν). Thus

$$G_1(x)=\lim_{a\to0+} c\,G_1(x)/\mathbf{K}(ax).$$

If for every $a>0$ $G_1(x)/\mathbf{K}(ax)$ is non-decreasing so is $G_1(x)$ but this is impossible because $G_1\in L^1(0,\infty)$.

Case ii. Here $\lim\limits_{x\to0+}\left(\log\frac{1}{x}\right)^{-1}\mathbf{K}(x)=c>0$. Thus

$$G_1(x)=\lim_{a\to0+} c\log\left(\frac{1}{a}\right)G_1(x)/\mathbf{K}(ax).$$

If for every $a>0$ $G_1(x)/\mathbf{K}(ax)$ is non-decreasing then so is $G_1(x)$; but this is impossible.

Case iii. Here $\lim\limits_{x\to0+}x^{2\nu-1}\mathbf{K}(x)=c>0$. Thus

$$G_1(x)\,x^{2\nu-1}=\lim_{a\to0+} c\,a^{2\nu-1}G_1(x)/\mathbf{K}(ax),$$

etc.

6. Variation diminishing kernels.

In this section we will prove the converse of Theorem 4b.

Lemma 6a. *Let $G(x)$ be a variation diminishing kernel. If $f(x)$ is continuous for $0<x<\infty$ and is such that for each $y>0$ the integral*

$$G \# f \cdot (y) = \int_0^\infty \int_0^\infty G(x) f(z) D(x, y, z) d\mu(x) d\mu(z)$$

is absolutely convergent then

$$V[G \# f] \leqq V[f].$$

Proof. Let

$$f_n = \begin{cases} -n & \text{if} \quad f(x) \leqq -n \\ f(x) & \text{if} \quad -n < f(x) < n \\ n & \text{if} \quad f(x) \geqq n \;. \end{cases}$$

Then for each positive integer n, $f_n \in C$; thus

$$V[G \# f_n] \leqq V[f_n] = V[f].$$

By Lebesgue's convergence theorem for each $y > 0$

$$G \# f \cdot (y) = \lim_{n \to \infty} G \# f_n \cdot (y),$$

and from this it follows that

$$V[G \# f] \leqq \varliminf_{n \to \infty} V[G \# f_n],$$

etc.

We require the following result which is a special case of a theorem of Pólya, see Obrechkoff [9; p. 11].

Theorem 6b. *Let $\varphi(z)$ be analytic for $|z| \leqq r$ for some $r > 0$. If there exists a sequence of polynomials $P_n(z)$ of the form*

$$P_n(z) = \prod_{j=1}^{n} \left[1 - \left(\frac{z}{a_{nj}}\right)^2\right]$$

where the a_{nj} are positive, such that

$$\lim_{n \to \infty} P_n(z) = \varphi(z)$$

uniformly in the circle $|z| \leqq r$, then $\varphi(z)$ is of the form

$$\varphi(z) = e^{-cz^2} \prod_j \left[1 - \left(\frac{z}{a_j}\right)^2\right]$$

where c is non-negative, the a_j are positive, and $\Sigma_j a_j^{-2} < \infty$.

Theorem 6c. *If $G(x)$ is a variation diminishing density function then*

$$G^{\wedge}(t) = 1 / E(t)$$

where

$$E(t) = e^{ct^2} \prod_k \left[1 + \frac{t^2}{a_k}\right].$$

Here $c \geqq 0$, $a_k > 0$, $k = 1, 2, \ldots$, $\Sigma_k 1/a_k^2 < \infty$.

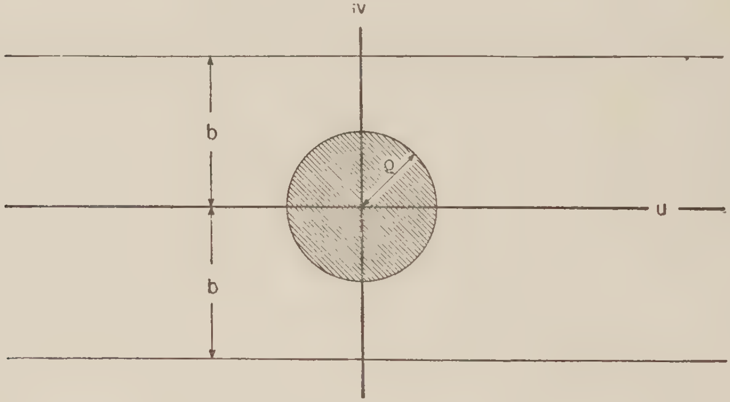
Proof. Let G_1 be defined as in § 5, and let

$$G_1^{\wedge}(t) = \int_0^{\infty} G_1(x)\, \mathbf{J}(xt)\, d\mu(x). \tag{1}$$

Theorem 5d implies that for some constant $b > 0$

$$0 \leqq G_1(x) \leqq A\, e^{-bx} \qquad 0 < x < \infty. \tag{2}$$

Let us permit t in (1) to take on complex values, $t = u + iv$. Using (2) and familiar estimates for Bessel functions in the complex plane we find



that $G_1^{\wedge}(t)$ is analytic in the strip $|\operatorname{Im} t| < b$. Because $G_1(x)$ is a density function $G_1^{\wedge}(0) = 1$. It follows that if

$$\Omega(t) = 1 / G_1^{\wedge}(t)$$

then $\Omega(t)$ is analytic in some circle $|t| < \rho$ about $t = 0$. Moreover it is apparent from the evenness of $\mathbf{J}(xt)$ that $\Omega(t)$ is even. Thus we have

$$\Omega(t) = \sum_0^{\infty} \omega_k t^{2k} \qquad |t| < \rho .$$

If we multiply both sides of the formula

$$\int_0^{\infty} \mathbf{J}(zt) D(x, y, z) \, d\mu(z) = \mathbf{J}(xt) \mathbf{J}(yt)$$

by $G_1(x) \, d\mu(x)$ and integrate we find that

$$\int_0^{\infty} \int_0^{\infty} G_1(x) \mathbf{J}(zt) D(x, y, z) \, d\mu(z) \, d\mu(x) = G_1^{\wedge}(t) \mathbf{J}(yt)$$

or

$$(1) \qquad \int_0^{\infty} \int_0^{\infty} G_1(x) \Omega(t) \mathbf{J}(zt) D(x, y, z) \, d\mu(z) \, d\mu(x) = \mathbf{J}(yt) .$$

For positive ε and integral n let us set

$$p_{\varepsilon, n}(r) = (\varepsilon - r)(2\varepsilon - r) \dots (n\varepsilon - r) .$$

Let $\triangle$ be defined as in § 2, the differentiation being with respect to t. Applying $p_{\varepsilon, n}(-\triangle)$ to both sides of (1) we obtain formally

$$(2) \qquad \int_0^{\infty} \int_0^{\infty} G_1(x) \{p_{\varepsilon, n}(-\triangle) \Omega(t) \mathbf{J}(zt)\} D(x, y, z) \, d\mu(z) \, d\mu(x)$$

$$= p_{\varepsilon, n}(-\triangle) \mathbf{J}(yt)$$

$$= p_{\varepsilon, n}(y^2) \mathbf{J}(yt) .$$

Here we have used (9) from § 2. The differentiation under the integral sign may be justified by the argument in § 3. Let us now set $t = 0$ in (2). Then if we let

$$q_{\varepsilon, n}(z) = \{p_{\varepsilon, n}(-\triangle) \, \Omega(t) \mathbf{J}(zt)\}_{t=0}$$

we obtain

$$\int_0^{\infty} \int_0^{\infty} G_1(x) q_{\varepsilon, n}(z) D(x, y, z) \, d\mu(x) \, d\mu(z) = p_{\varepsilon, n}(y^2) .$$

By Lemma 6a

$$n = V\,[p_{\varepsilon,n}(y^2)] \leq V\,[q_{\varepsilon,n}(z)]$$

from which it follows that $q_{\varepsilon,n}(z)$ which is an even polynomial of degree $2n$ in z has only real zeros. Letting $\varepsilon \to 0$ it follows from a theorem of Hurwitz, see [14; 3. 45], that $q_n(z) = q_{0,n}(z)$ has only real zeros. We have

$$q_n(z) = \{\triangle^n\, \Omega(t)\, \mathbf{J}(zt)\}_{t=0}\,.$$

Since

$$\triangle t^{2m} = 2m\,(2m-1+2\nu)\,t^{2m-2}$$

it follows that

$$q_n(z) = 2^{2n}\, n!\, (\nu+1/2)_n \sum_{k=0}^{n} \omega_{n-k} \frac{(-1)^k}{k!\,(\nu+1/2)_k} \left(\frac{z}{2}\right)^{2k}.$$

Here

$$(\alpha)_r = \alpha\,(\alpha+1)\ldots(\alpha+r-1)\,.$$

We now define

$$\begin{aligned} Q_n(w) &= (-1)^n \left(\frac{w}{2n}\right)^{2n} q_n\left(\frac{2n}{w}\right) \\ &= \sum_{k=0}^{n} (-1)^k\, \omega_k \left[\frac{n!}{(n-k)!\, n^k}\right] \left[\frac{(\nu+1/2)_n}{(\nu+1/2)_{n-k}\, n^k}\right] w^{2k}\,. \end{aligned}$$

$Q_n(w)$ has only real zeros. It is easily seen that

$$\lim_{n\to\infty} Q_n(w) = \sum_{k=0}^{\infty} (-1)^k\, \omega_k\, w^{2k}$$

uniformly for $|w| \leq r$ if $r < \rho$. By Theorem 6b

$$\sum_{k=0}^{\infty} (-1)^k\, \omega_k\, w^{2k} = e^{-c_1 w^2} \prod_k \left[1 - \frac{w^2}{a_k^2}\right]$$

where

$$c_1 \geq 0\,, \quad 0 < a_1 \leq a_2, \ldots, \quad \sum_k a_k^{-2} < \infty\,.$$

Replacing w by it we see that

$$G_1^{\wedge}(t) = \left[e^{c_1 t^2} \prod_k \left(1 + \frac{t^2}{a_k^2}\right)\right]^{-1}.$$

Now

$$G^{\wedge}(t)\, e^{-t^2} = G_1^{\wedge}(t)$$

so that

$$G^{\wedge}(t) = \left[e^{ct^2} \prod_k \left(1 + \frac{t^2}{a_k^2}\right)\right]^{-1}$$

where $c = c_1 - 1$. Since $G^{\wedge}(t)$ is necessarily bounded it follows that $c \geqq 0$. Our proof is now complete.

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VARIATION DIMINISHING TRANSFORMATIONS AND ULTRASPHERICAL POLYNOMIALS (*)

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1. Introduction.

Let φ be a real valued function defined on J the additive group of integers. We denote by $V[\varphi]$ the number of changes of sign of $\varphi(n)$ for $-\infty < n < \infty$. A real function G on J is said to belong to $l^1(J)$ if

$$\|G\|_1 = \sum_{-\infty}^{\infty} |G(n)|$$

is finite. $G \in l^1(J)$ is said to be variation diminishing on J if

$$V[G * \varphi] \leqq V[\varphi]$$

for every bounded real $\varphi \in l^\infty(J)$. Here

$$G * \varphi \cdot (n) = \sum_{k=-\infty}^{\infty} G(n-k)\,\varphi(k).$$

In 1952 Edrei [5] proved a conjecture of I. J. Schoenberg to the effect that G is variation diminishing on J if and only if (after multiplication by a suitable constant)

$$(1) \qquad G^\wedge(z) = z^m \exp[\varepsilon_1 z + \varepsilon_{-1} z^{-1}] \frac{\prod_{k=1}^{\infty}(1+\alpha_k z) \prod_{k=1}^{\infty}(1+\beta_k z^{-1})}{\prod_{k=1}^{\infty}(1-\gamma_k z) \prod_{k=1}^{\infty}(1-\delta_k z^{-1})}$$

where m is an integer, $\varepsilon_1 \geqq 0$, $\varepsilon_{-1} \geqq 0$, $\alpha_k \geqq 0$, $\beta_k \geqq 0$,

$$1 > \gamma_k \geqq 0, \quad 1 > \delta_k \geqq 0, \quad \text{and} \quad \Sigma_k (\alpha_k + \beta_k + \gamma_k + \delta_k) < \infty.$$

In the above formula

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(2) $$G^{\wedge}(z) = \sum_{-\infty}^{\infty} G(n) z^n,$$

and (1) holds in some annulus $\rho_1 < |z| < \rho_2$ where $\rho_1 < 1 < \rho_2$. Edrei's work completed partial results obtained earlier by Schoenberg [14], Aissen, Schoenberg and Whitney [1] and Edrei [4]. The demonstration of (1) is difficult and depends upon a number of very surprising arguments.

Let $p_n(x)$ $n = 0, 1, \ldots$ be the Legendre polynomials [6; vol. II, Chapter X]. We have

$$\int_{-1}^{1} p_n(x) p_m(x)\, dx = \frac{\delta_{n,m}}{n + 1/2}.$$

Let l^1 denote the class of those real functions $\varphi(n)$ defined for $n=0, 1, 2, \ldots$ for which

$$\|\varphi\|_1 = \sum_{n=0}^{\infty} |\varphi(n)| (n + 1/2)$$

is finite. For $\varphi \in l^1$ we set

$$\varphi^{\wedge}(x) = \sum_{n=0}^{\infty} \varphi(n) p_n(x) (n + 1/2).$$

Since, if $-1 \leqq x \leqq 1$,

$$|p_n(x)| \leqq 1 \qquad n = 0, 1, 2, \ldots,$$

it follows that $\varphi^{\wedge}(x)$ is continuous for $-1 \leqq x \leqq 1$. $\varphi(n)$ can be recaptured from $\varphi^{\wedge}(x)$ by means of the inversion formula

$$\varphi(n) = \int_{-1}^{1} \varphi^{\wedge}(x) p^n(x)\, dx \qquad n = 0, 1, \ldots.$$

Let us set

$$C(n, m, k) = \int_{-1}^{1} p_n(x) p_m(x) p_k(x)\, dx.$$

Then

(3) $$p_n(x) p_m(x) = \sum_{k=0}^{\infty} C(n, m, k) p_k(x) (k + 1/2).$$

(Here $C(n, m, k)$ vanishes for $k > n + m$). An explicit formula for $C(n, m, k)$ is given in § 2 which shows that

$$C(n, m, k) \geqq 0 . \tag{4}$$

If φ, ψ are functions defined for $n = 0, 1, \ldots$ we set

$$\varphi \# \psi \cdot (n) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \varphi(m) \psi(k) C(n, m, k) (m + 1/2)(k + 1/2) . \tag{5}$$

Evidently $\varphi \# \psi$ is well defined for $n = 0, 1, \ldots$. Setting $x = 1$ in (3) we find, since $p_n(1) = 1$ for $n = 0, 1, \ldots$, that

$$\sum_{k=0}^{\infty} C(n, m, k)(k + 1/2) = 1 ;$$

from this it is easily deduced that if $\varphi, \psi \in l^1$ then $\varphi \# \psi \in l^1$ and

$$\|\varphi \# \psi\|_1 \leqq \|\varphi\|_1 \cdot \|\psi\|_1 .$$

Moreover we have

$$(\varphi \# \psi)^{\wedge}(x) = \varphi^{\wedge}(x) \cdot \psi^{\wedge}(x) .$$

We thus find that under the convolution "$\#$" l^1 behaves quite like the group algebra $l^1(J)$. Let us denote by l^∞ the class of those real functions $\varphi(n)$, $n = 0, 1, \ldots$, for which

$$\|\varphi\|_\infty = \text{l.u.b.}\, |\varphi(n)|$$

is finite. If $G \in l^1$ and $\varphi \in l^\infty$ then $G \# \varphi$ may be defined by (5) and it may be checked that

$$\|G \# \varphi\|_\infty \leqq \|G\|_1 \|\varphi\|_\infty .$$

A function $G \in l^1$ will be called variation diminishing if

$$\mathrm{V}[G \# \varphi] \leqq \mathrm{V}[\varphi]$$

for every $\varphi \in l^\infty$. Here $\mathrm{V}[\varphi]$ is, of course, the number of changes of sign of $\varphi(n)$ for $n = 0, 1, 2, \ldots$. We shall show in the present paper that $G \in l^1$ is variation diminishing if and only if (after multiplication by a suitable constant)

$$G^{\wedge}(x) = \sum_{n=0}^{\infty} G(n) p_n(x)(n + 1/2) \tag{6}$$

$$= \exp cx \frac{\prod_{k=1}^{\infty}(1 + a_k x)}{\prod_{k=1}^{\infty}(1 - b_k x)}$$

where $c \geqq 0$, $1 \geqq a_k \geqq 0$, $1 > b_k \geqq 0$, and $\Sigma_k a_k + b_k < \infty$. Actually our results apply to ultraspherical polynomials and we have confined ourselves to Legendre polynomials in this section merely to avoid the introduction of cumbersome notations. Analogous results applying to Hankel transforms have been obtained in the previous paper [8]. The argument there paralleled the argument given by Schoenberg for variation diminishing functions in the group algebra of the line. The argument here is different in that after various reductions (6) will be obtained as a consequence of (1).

2. Ultraspherical polynomials.

In this section we have listed the properties of ultraspherical polynomials which we will need. For a more detailed and more motivated discussion of the harmonic analysis of ultraspherical polynomials we refer the reader to [7]. For a fixed value [1] of $\nu > 0$ we set

$$2^n (\nu + 1/2)_n W(n, x) = (-1)^n (1 - x^2)^{1/2 - \nu} \left(\frac{d}{dx}\right)^n (1 - x^2)^{n + \nu - 1/2}.$$

The $W_\nu(n, x)$ are the ultraspherical polynomials of index ν normalized by the condition

$$W_\nu(n, 1) = 1 \qquad n = 0, 1, \ldots .$$

We therefore have

$$W_\nu(n, x) = C_n^\nu(x) \frac{n!}{(2\nu)_n}$$

where the $C_n^\nu(x)$ are the ultraspherical polynomials normalized in the usual way, see [6; vol. II, Chapter X]. If

$$d\Omega_\nu(x) = (1 - x^2)^{\nu - 1/2} dx, \tag{1}$$

$$\omega_\nu(n) = \frac{\Gamma(\nu)(2\nu)_n (n + \nu)}{\pi^{1/2}\Gamma(\nu + 1/2)\, n!},$$

1. The case $\nu = 0$ is Edrei's theorem for an even kernel on J.

then

(2) $$\int_{-1}^{1} W_\nu(n, x) W_\nu(m, x)\, d\Omega_\nu(x) = \delta_{n,m} / \omega_\nu(n).$$

Let us denote by l_ν^1 the linear space consisting of those real functions φ on $n = 0, 1, 2, \ldots$ for which

$$\|\varphi\|_1 = \sum_{n=0}^{\infty} |\varphi(n)|\, \omega_\nu(n)$$

is finite. For $\varphi \in l_\nu^1$ we set

(3) $$\varphi^\wedge(x) = \sum_{n=0}^{\infty} \varphi(n)\, \omega_\nu(n)\, W_\nu(n, x).$$

Clearly $\varphi^\wedge(x)$ is a bounded continuous function on $-1 \leqq x \leqq 1$ since if $|x| \leqq 1$ then $|W_\nu(n, x)| \leqq 1$, $n = 0, 1, 2, \ldots$. For $\varphi \in l_\nu^1$ we have the inversion formula

(4) $$\varphi(n) = \int_{-1}^{1} \varphi^\wedge(x)\, W_\nu(n, x)\, d\Omega_\nu(x).$$

We require the following formula of Dougall, see Hsü [10],

(5) $$W_\nu(k, x) W_\nu(j, x) = \sum_{n=0}^{\infty} C_\nu(k, j, n)\, W_\nu(n, x)\, \omega_\nu(n).$$

Here $C_\nu(k, j, n)$ is zero unless $k + j + n = 2\sigma$ is even and unless $\max(k, j, n) \leqq \sigma$. If these conditions are met then

(6) $$C_\nu(k, j, n) = \frac{\pi\, \Gamma(2\nu)\, 2^{1-2\nu}}{\Gamma(\nu)^2} \cdot \frac{(1)_k (1)_j (1)_n}{(2\nu)_k (2\nu)_j (2\nu)_n}$$
$$\cdot \frac{(\nu)_{\sigma-k} (\nu)_{\sigma-j} (\nu)_{\sigma-n}}{(1)_{\sigma-k} (1)_{\sigma-j} (1)_{\sigma-n}} \cdot \frac{(2\nu)_\sigma}{(\nu)_\sigma} \cdot \frac{1}{\sigma+\nu}.$$

For $\varphi(n)$, $\psi(n)$ defined for $n = 0, 1, \ldots$ we set

(7) $$\varphi \# \psi \cdot (n) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \varphi(k)\, \psi(j)\, C_\nu(k, j, n)\, \omega_\nu(k)\, \omega_\nu(j).$$

Note that for each n only a finite number of terms on the right hand side of (7) are different from zero. Since

$$C_\nu(k, j, n) \geqq 0, \tag{8}$$

$$\sum_{n=0}^{\infty} C_\nu(k, j, n)\, \omega_\nu(n) = 1, \tag{9}$$

(here (8) follows from (6) and (9) follows from (5) if x is set equal to 1) we find that

$$\|\varphi \# \psi\|_1 \leqq \|\varphi\|_1 \cdot \|\psi\|_1. \tag{10}$$

Thus if $\varphi, \psi \in l_\nu^1$ so is $\varphi \# \psi$ and using (5) it is easily verified that

$$(\varphi \# \psi)\hat{} = \varphi\hat{} \cdot \psi\hat{}. \tag{11}$$

It is also easy to show using (8) and (9) that

$$\|\varphi \# \psi\|_\infty \leqq \|\varphi\|_1 \|\psi\|_\infty$$

where $\|\psi\|_\infty = \text{l.u.b.}\, |\psi(n)|$ for $n = 0, 1, \ldots$. We denote by l_ν^∞ the set of all (real) functions $\varphi(n)$ for which $\|\psi\|_\infty < \infty$. Let $\varphi, \psi, \chi \in l_\nu^1$ then

$$[(\varphi \# \psi) \# \chi]\hat{} = (\varphi\hat{} \cdot \psi\hat{})\chi\hat{} = \varphi\hat{}(\psi\hat{} \cdot \chi\hat{}) = [\varphi \# (\psi \# \chi)]\hat{}.$$

The inversion formula (4) now implies that

$$(\varphi \# \psi) \# \chi = \varphi \# (\psi \# \chi)$$

if $\varphi, \psi, \chi \in l_\nu^1$. A simple approximation argument shows that the above is also valid if $\varphi, \psi \in l_\nu^1$, $\chi \in l_\nu^\infty$.

We will have occasion to use the recursion formula, valid for $k = 0, 1, 2, \ldots$,

$$x W_\nu(k, x) = \frac{k}{2k + 2\nu} W_\nu(k - 1, x) + \frac{k + 2\nu}{2k + 2\nu} W_\nu(k + 1, x). \tag{12}$$

Here if $k = 0$ the term $W_\nu(-1, x)$ on the right must be taken as 0. This can be obtained as the special case of (5) when $j = 1$.

We will also have occasion to use ultraspherical functions of the second kind. We set

$$V_\nu(n, x) = \int_{-1}^{1} (x - t)^{-1} W_\nu(n, t)\, d\Omega_\nu(t). \tag{13}$$

The relation of $V_\nu(n, n)$ to the standard function of the second kind is

$$V_\nu(n, x) = Q_n^{(\nu-1/2,\, \nu-1/2)}(x)\,(x^2 - 1)^{\nu-1/2} \frac{n! \cdot 2}{(\nu + 1/2)_n}.$$

Starting from (12) it is easily shown that

$$(14) \qquad x V_\nu(k, x) = \frac{k}{2k+2\nu} V_\nu(k-1, x) + \frac{k+2\nu}{2k+2\nu} V_\nu(k+1, x)$$

for $k = 1, 2, \ldots$. For $k = 0$ we have

$$(14') \qquad x V_\nu(0, x) = V_\nu(1, x) + \int_{-1}^{1} d\Omega_\nu(t).$$

An alternative integral representation for $V_\nu(n, x)$, see [15; § 4.61], is

$$(15) \qquad V_\nu(n, x) = \frac{2^{-n} n!}{(\nu + 1/2)_n} \int_{-1}^{1} (1-t^2)^n (x-t)^{-n-1} d\Omega_\nu(t).$$

It is easily deduced from (15) that if $x > 1$ then

$$(16) \qquad \log V_\nu(n, x) = n \log (x - \sqrt{x^2-1}) + O(\log n)$$

as $n \to \infty$. In particular if $x > 1$ is fixed then $V_\nu(n, x)$ decreases exponentially as $n \to \infty$. We also need to know how $V_\nu(n, x)$ behaves as $x \to 1+$. There are three cases. If $0 < \nu < 1/2$ then it follows from (13) that

$$V_\nu(n, x) = V_\nu(0, x) = \int_{-1}^{1} (x-t)^{-1} \{W_\nu(n, t) - 1\} d\Omega_\nu(t)$$
$$= O(1) \qquad (x \to 1+).$$

On the other hand [2]

$$V_\nu(0, x) = \int_{-1}^{1} (x-t)^{-1} d\Omega_\nu(t) \cong (x-1)^{\nu - 1/2} 2^{\nu - 1/2} \pi \sec \pi\nu$$

as $x \to 1+$. Thus for $n = 0, 1, 2, \ldots$ we have

$$(17) \qquad V_\nu(n, x) \cong (x-1)^{\nu-1/2} 2^{\nu-1/2} \pi \sec \pi\nu \quad \text{as} \quad x \to 1+ \quad (0 < \nu < 1/2).$$

An entirely similar argument shows that for $n = 0, 1, \ldots$ we have

$$(17') \qquad V_{1/2}(n, x) \cong \log (x-1)^{-1} \qquad \text{as } x \to 1+$$

2. $A(x) \cong B(x)$ as $x \to 1+$ means $\lim_{x \to 1+} A(x)/B(x) = 1$.

If $\nu > 1/2$ then for $n = 0, 1, \dots$ we have

(17″) $$V_\nu(n, 1+) = 2^{2\nu-1}\,\Gamma(\nu - 1/2)\,\Gamma(\nu + 1/2)\,\frac{n!}{\Gamma(n + 2\nu)}\,.$$

This follows from [15; 4.62.5].

3. Elementary kernels. (3)

A function $f(n)$ defined for $n = 0, 1, \dots$ is said to have at least m changes of sign if there exist integers

$$0 \leqq n_0 < n_1 < \dots < n_m$$

such that

$$f(n_k)\, f(n_{k-1}) < 0 \qquad k = 1, 2, \dots, m\,.$$

f has exactly m changes of sign if it has at least m changes of sign and does not have at least $m + 1$ changes of sign. The number of changes of sign of f is denoted by $\mathrm{V}[f]$; $\mathrm{V}[f]$ has one of the values $0, 1, \dots, \infty$.

Definition 3a. *A function $K(n) \in l^1$ is said to be a variation diminishing kernel if for every $f \in l^\infty$*

$$\mathrm{V}[K \# f] \leqq \mathrm{V}[f]\,.$$

In the present section we will exhibit two families of simple variation diminishing kernels. Let us set

$$r_-(n) = \frac{n}{2n + 2\nu}\,\frac{\omega(n)}{\omega(n-1)} \qquad r_+(n) = \frac{n + 2\nu}{2n + 2\nu}\,\frac{\omega(n)}{\omega(n+1)}\,.$$

We then have, see (12) § 2,

(1) $$x\,\omega(n)\,W(n, x) = r_-(n)\,\omega(n-1)\,W(n-1, x) + r_+(n)\,\omega(n+1)\,W(n+1, x)\,.$$

We define the operator $\triangle$ by the formula

(2) $$\triangle f \cdot (n) = r_+(n-1)\,f(n-1) + r_-(n+1)\,f(n+1) \qquad n = 0, 1, \dots\,.$$

Here $r_+(-1)$ is to be taken as 0. Let I be the identity operator

(3) $$I f \cdot (n) = f(n) \qquad n = 0, 1, \dots\,.$$

3. In this section and those that follow we shall usually drop the subscript ν. The reader should imagine that a fixed value of ν has been chosen and that this value is used throughout.

Lemma 3b. If $\xi > 1$ then

$$\mathrm{V}\,[(\triangle - \xi I)\,\varphi] \geqq \mathrm{V}\,[\varphi]$$

for every $\varphi \in l^{\infty}$.

To prove this let us introduce the operators

$$\delta_+ f(n) = f(n+1) - f(n) \qquad n = 0, 1, \ldots,$$
$$\delta_- f(n) = f(n) - f(n-1) \qquad n = 0, 1, \ldots;$$

here $f(-1)$ is to be taken as 0. We further define the functions

$$\alpha(n) = \frac{1}{\omega(n)\, W(n\,,\,\xi)}\,,$$

$$\beta(n) = \frac{(2\nu+1)_n}{n\,!}\, W(n\,,\,\xi)\, W(n+1\,,\,\xi)\,,$$

$$\gamma(n) = \frac{\omega(n)\, n\,!}{(2\nu+1)_n}\,\frac{n+2\nu}{2n+2\nu}\,[W(n\,,\,\xi)]^{-1}.$$

We assert that

(4) $$\triangle - \xi I = (\alpha I)\,\delta_-\,(\beta I)\,\delta_+\,(\gamma I)\,.$$

It is evident that

$$\begin{aligned}(\alpha I)\,\delta_-\,(\beta I)\,\delta_+\,(\gamma I\,\varphi) &= \varphi(n+1)\,[\alpha(n)\,\beta(n)\,\gamma(n+1)]\\ &\quad - \varphi(n)\,[\alpha(n)\,\beta(n)\,\gamma(n) + \alpha(n)\,\beta(n-1)\,\gamma(n)]\\ &\quad + \varphi(n-1)\,[\alpha(n)\,\beta(n-1)\,\gamma(n-1)]\end{aligned}$$

so that proving (4) amounts to verifying that for $n = 0, 1, \ldots$

$$\alpha(n)\,\beta(n)\,\gamma(n+1) = r_-(n+1)\,,$$
$$\alpha(n)\,\beta(n-1)\,\gamma(n-1) = r_+(n-1)\,,$$
$$\alpha(n)\,\beta(n)\,\gamma(n) + \alpha(n)\,\beta(n-1)\,\gamma(n) = \xi\,.$$

Using the definitions of α, β and γ and (1) the above identities are easily checked. Simple geometric considerations show that if $\zeta(n) > 0$ $n = 0, 1, 2, \ldots$ then for every $\varphi \in l^{\infty}$

$$\mathrm{V}\,[\delta_-\,(\zeta I\,\varphi)] \geqq \mathrm{V}\,[\varphi]\,,$$

while if $\eta(n) > 0$, $n = 0, 1, 2, \ldots$ and if $\eta(n) \to 0$ as $n \to \infty$ then

$$\mathrm{V}\,[\delta_+\,(\eta I\,\varphi)] \geqq \mathrm{V}\,[\varphi]\,.$$

Since the zeros of $W(n\,,\,x)$ lie in $-1 < x < 1$ and since $W(n\,,\,1) = 1$

it follows that $W(n, \xi) > 0$ if $\xi > 1$. Thus $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ are > 0 for $n = 0, 1, \ldots$. We also see [15; Theorem 8.21.10] that $\gamma(n) \to 0$ as $n \to \infty$.

Lemma 3c. *If* $\varphi \in l^1$ *then*

$$[(\triangle - \xi I)\varphi]^\wedge(x) = (x - \xi)\varphi^\wedge(x).$$

The boundedness as $n \to \infty$ of $r_-(n)$ and $r_+(n)$ implies that $(\triangle - \xi I)\varphi \in l^1$ whenever $\varphi \in l^1$. Direct computation using (1) gives

$$\begin{aligned}
[(\triangle - \xi I)\varphi]^\wedge(x) \\
&= \sum_{n=0}^{\infty} [r_-(n+1)\varphi(n+1) - \xi\varphi(n) + r_+(n-1)\varphi(n-1)]\,\omega(n)W(n, x) \\
&= \sum_{n=0}^{\infty} \varphi(n)[r_-(n)\omega(n-1)W(n-1, x) + r_+(n)\omega(n+1)W(n+1, x) \\
&\qquad\qquad - \xi\omega(n)W(n, x)] \\
&= \sum_{n=0}^{\infty} \varphi(n)\omega(n)W(n, x)(x-\xi) = (x-\xi)\varphi^\wedge(x).
\end{aligned}$$

Let $V(n, \xi) = V_\nu(n, \xi)$ be defined as in § 2. It follows from (16) of § 2 that $V(n, \xi) \in l^1_\nu$. The formula (13) § 2 then implies that

$$V(\cdot, \xi)^\wedge(x) = (\xi - x)^{-1}. \tag{5}$$

Theorem 3d. *For* $\xi > 1$ $V(n, \xi)$ *is a variation diminishing kernel in* l^1.

Using (5). Lemma 3c and (11) of § 2 we see that

$$[(\triangle - \xi I)(V(\cdot, \xi) \# \varphi)]^\wedge = \varphi^\wedge$$

and thus that

$$(\triangle - \xi I)(V(\cdot, \xi) \# \varphi) = \varphi.$$

By Lemma 3b

$$\mathbf{V}[V(\cdot, \xi) \# \varphi] \leqq \mathbf{V}[\varphi],$$

as desired.

Lemma 3e. *If* $\xi > +1$ *then*

$$\mathbf{V}[(\triangle + \xi I)\varphi] \leqq \mathbf{V}[\varphi]$$

for every $\varphi \in l^\infty$.

We introduce the operators

$$\sigma_+ f(n) = f(n+1) + f(n) \qquad n = 0, 1, \dots,$$
$$\sigma_- f(n) = f(n) + f(n-1) \qquad n = 0, 1, \dots;$$

here again $f(-1)$ is to be taken as 0. We assert that if $\alpha(n)$, $\beta(n)$, $\gamma(n)$ are defined as in the proof of Lemma 3b then

$$(\triangle + \xi I) = (\alpha I)\, \sigma_- (\beta I)\, \sigma_+ (\gamma I).$$

The verification can be carried out almost exactly as before. If $\zeta(n) > 0$ $n = 0, 1, 2, \dots$ then for every $\varphi \in l^\infty$

$$V[\sigma_+ (\zeta I)\varphi] \leqq V[\varphi],$$
$$V[\sigma_- \zeta(I)\varphi] \leqq V[\varphi].$$

See Pólya and Szegö, [11; vol. 2, p. 38]. Our lemma is an immediate consequence of these assertions.

Let us define

$$U(n, \xi) = \begin{cases} \xi / \omega(0) & n = 0 \\ 1/\omega(1) & n = 1 \\ 0 & n > 1. \end{cases}$$

Theorem 3f. *If $\xi > 1$ then $U(n, \xi)$ is a variation diminishing kernel in l^1.*

This follows on noting that

$$U(\cdot, \xi)^\wedge(x) = (\xi + x)$$

and thus that

$$[U(\cdot, \xi) \# \varphi]^\wedge(x) = [(\triangle + \xi I)\varphi]^\wedge(x)$$

so that

$$U(\cdot, \xi) \# \varphi \cdot (n) = (\triangle + \xi I)\varphi \cdot (n).$$

4. Composite kernels.

In this section we will use the simple variation diminishing kernels found in the preceding section to build up more complex variation diminishing kernels.

Lemma 4a. *If $K_1(n)$ and $K_2(n)$ are variation diminishing kernels in l^1 then so is $K_1 \# K_2(n)$.*

Let $\varphi \in l^\infty$ then

$$V[(K_1 \# K_2) \# \varphi] \leqq V[K_2 \# \varphi] \leqq V[\varphi].$$

Theorem 4b. Let $c > 0$,

$$1 \geqq a_1 \geqq a_2 \geqq \ldots \geqq 0, \quad \text{and} \quad 1 > b_1 \geqq b_2 \geqq \ldots \geqq 0,$$

where

$$\Sigma_k (a_k + b_k) < \infty.$$

If

$$E(x) = e^{cx} \cdot \frac{\Pi_k (1 + a_k x)}{\Pi_k (1 - b_k x)}$$

then there is a (unique) $G(n) \in l^1$ such that $G^\wedge(x) = E(x)$, and $G(n)$ is variation diminishing.

If $E(x)$ is of the form

$$\prod_{k=1}^{r} (1 + a_k x)$$

then the theorem is true with

$$G(\cdot) = a_1 U(\cdot, a_1^{-1}) \# \ldots \# a_r U(\cdot, a_r^{-1}).$$

Consider next the case

$$E(x) = \prod_{k=1}^{\infty} (1 + a_k x).$$

Let us define

$$G(n) = \int_{-1}^{1} E(x) W(n, x) d\Omega(x).$$

We set

$$G_r(n) = a_1 U(\cdot, a_1^{-1}) \# \ldots \# a_r U(\cdot, a_r^{-1}).$$

Then

$$G_r(n) = \int_{-1}^{1} E_r(x) W(n, x) d\Omega(x)$$

where

$$E_r(x) = \prod_{k=1}^{r} (1 + a_k x).$$

Thus

$$G(n) - G_r(n) = \int_{-1}^{1} [E(x) - E_r(x)]\, W(n, x)\, d\Omega(x).$$

Since $E_r(x) \to E(x)$ uniformly on $-1 \leqq x \leqq 1$ as $r \to \infty$ it follows that

(1) $$\lim_{r\to\infty} G_r(n) = G(n) \qquad n = 0, 1, \ldots.$$

The functions $a_k U(n, a_k^{-1})$ are non-negative, and therefore so is $G_r(n)$. It follows that $G(n)$ too is non-negative. Now

$$\| G_r(n) \|_1 = \sum_{n=0} G_r(n)\, \omega(n) = G^\wedge(1) = \prod_{k=1}^{r} (1 + a_k).$$

By what is essentially Fatou's lemma we find that

$$\| G(n) \|_1 \leqq \varliminf_{n\to\infty} \| G_r(n) \|_1 = \prod_{k=1}^{\infty} (1 + a_k).$$

Thus $G \in l^1$, and evidently $G^\wedge(x) = E(x)$. It remains only to prove that G is variation diminishing. If $\varphi \in l^1$ then by (1)

$$G \# \varphi \cdot (n) = \lim_{r\to\infty} G_r \# \varphi \cdot (n) \qquad n = 0, 1, \ldots$$

and thus

$$\mathrm{V}[G \# \varphi] \leq \varliminf_{r\to\infty} \mathrm{V}[G_r \# \varphi].$$

But since G_r is known to be variation diminishing we have

$$\mathrm{V}[G_r \# \varphi] \leqq \mathrm{V}[\varphi]$$

for each r and thus

$$\mathrm{V}[G \# \varphi] \leqq \mathrm{V}[\varphi].$$

An evident approximation argument enables us to show that this is valid for $\varphi \in l^\infty$ and not merely for $\varphi \in l^1$.

If $E(x)$ is of the form

$$E(x) = 1 \Big/ \prod_{k=1}^{r} (1 - b_k x)$$

then the theorem is true with

$$G(\cdot) = b_1 V(\cdot, b_1^{-1}) \# \ldots \# b_r V(\cdot, b_r^{-1}).$$

An argument almost exactly like the one just given shows that our theorem is true if

$$E(x) = 1 \Big/ \prod_{k=1}^{\infty} (1 - b_k x).$$

Finally if

$$E(x) = e^{cx} \qquad c > 0$$

we set

$$E_r(x) = \left(1 + \frac{xc}{n}\right)^n$$

or

$$E_r(x) = 1 \Big/ \left(1 - \frac{xc}{n}\right)^n.$$

In either case $E_r(x) \to E(x)$ uniformly on $-1 \leq x \leq 1$, etc.

5. An order property.

In this section we will show that if $G(j)$ is a variation diminishing kernel in l_v^1 then $G(j)$ vanishes exponentially as $j \to \infty$. This information is needed later.

Lemma 5a. *If $G(j)$ is a variation diminishing kernel in l_v^1 then* $\mathrm{V}[G(j)] = 0$.

This, of course, means that one of the relations $G(j) \geq 0$, $j = 0, 1, \ldots$ or $G(j) \leq 0$, $j = 0, 1, \ldots$ must hold. Let

$$\eta(n) = \begin{cases} 1/\omega_v(0) & n = 0 \\ 0 & n > 0. \end{cases}$$

Then $\eta(n) \in l_v^1$ and $\eta^\wedge(x) = 1$. It follows that for any $\varphi \in l_v^1$ $\eta \# \varphi = \varphi$.

Using this and the fact G is variation diminishing we have

$$\mathbf{V}[G] = \mathbf{V}[G \# \eta] \leqq \mathbf{V}[\eta] = 0 .$$

Since we can replace $G(j)$ by $-G(j)$, if we wish, we see that it is no restriction to assume that $G(j) \geqq 0$, $j = 0, 1, \ldots$.

Let $\triangle$ be defined as in § 3, and let $\xi > 1$. We assert that

$$\triangle - \xi I = (\lambda I)\, \delta_+ (\varkappa I)\, \delta_- (\mu I) \tag{1}$$

where

$$\lambda(n) = \frac{1}{\omega(n)\, V(n, \xi)} \qquad n = 0, 1, \ldots,$$

$$\varkappa(n) = \frac{(2\nu+1)_{n-1}}{(n-1)!} V(n, \xi)\, V(n-1, \xi) \qquad n = 1, 2, \ldots,$$

$$\varkappa(0) = V(0, \xi) \cdot \int_{-1}^{1} d\Omega_\nu(t)$$

$$\mu(n) = \frac{\omega(n)\, n!}{(2\nu+1)_n} \, \frac{n+2\nu}{2n+2\nu} \, \frac{1}{V(n, \xi)} \qquad n = 0, 1, \ldots.$$

The verification of (1) runs parallel to the arguments of § 3. We have

$$\begin{aligned}(\lambda I)\, \delta_+ (\varkappa I)\, \delta_- (\mu I)\, \varphi = {} & \varphi(n+1)\, [\lambda(n)\, \varkappa(n+1)\, \mu(n+1)] \\ & - \varphi(n)\, [\lambda(n)\, \varkappa(n+1)\, \mu(n) + \lambda(n)\, \varkappa(n)\, \mu(n)] \\ & + \varphi(n-1)\, [\lambda(n)\, \varkappa(n)\, \mu(n-1)] .\end{aligned}$$

Thus we must verify that for $n = 0, 1, \ldots,$

$$\lambda(n)\, \varkappa(n+1)\, \mu(n+1) = r_-(n+1) ,$$

$$\lambda(n)\, \varkappa(n)\, \mu(n-1) = r_+(n-1) ,$$

$$\lambda(n)\, \varkappa(n+1)\, \mu(n) + \lambda(n)\, \varkappa(n)\, \mu(n) = \xi .$$

The first two of these relations are easily checked. The last relation must be checked separately for $n \geqq 1$ and for $n = 0$, see (14) and (14') of § 2.

Lemma 5b. *Let $G(j)$ be a non-negative variation diminishing kernel in l^1_ν and let $\xi > 1$. Then*

$$G(j)\, \{\omega(j)\, V(j, \xi)\}^{-1}$$

cannot have a local minimum.

By a local minimum we mean an integer $k > 0$ and that if

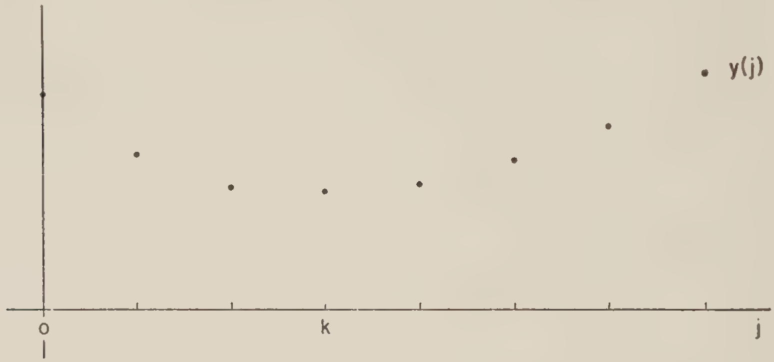
$$y(j) = G(j)\{\omega(j) V(j, \xi)\}^{-1}$$

then

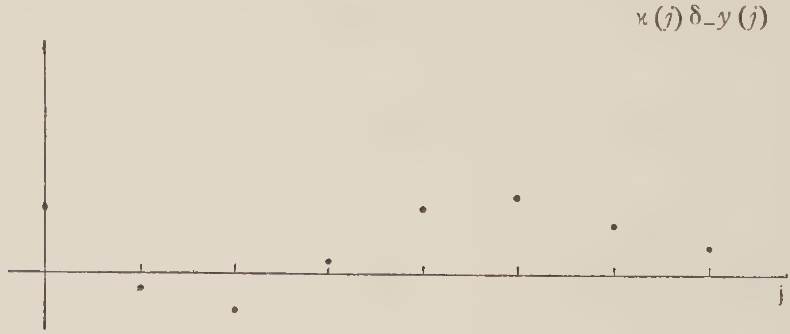
$$y(k) \leqq \begin{cases} y(k-1) \\ y(k+1), \end{cases}$$

$$y(k) < \begin{cases} \text{l.u.b. } [y(j) \mid j = 0, \ldots, k-1] \\ \text{l.u.b. } [y(j) \mid j = k+1, k+2, \ldots]. \end{cases}$$

If $y(j)$ had a local minimum it would appear as in the figure below.



This would then constrain $\varkappa(j)\, \delta_- y(j)$ to take on in succession positive



negative, and positive values. Since $V(j, \xi)$ vanishes exponentially as $j \to +\infty$, see (16) of § 2, it follows that $\varkappa(j)\, \delta_- y(j) \to 0$ as $j \to \infty$. From this it is evident that

$$\lambda\delta + \varkappa\delta_- y = (\triangle - \xi I) G$$

would have to have at least two changes of sign. Now it is easily verified that

$$(\Delta - \xi I) G = G \# (\Delta - \xi I) \eta .$$

Direct computation shows that

$$V[(\Delta - \xi I) \eta] = 1 ;$$

since G is variation diminishing, this implies that

$$V[(\Delta - \xi I) G] \leqq 1 .$$

These two facts taken in conjunction prove our lemma.

Theorem 5c. *If $G(j)$ is a non-negative variation diminishing kernel in l^1_ν then there is a number $\xi > 1$ such that*

$$G(j) = O[\omega_\nu(j) V_\nu(j , \xi)] \qquad j \to + \infty .$$

We know from Lemma 5b that for a given $\xi > 1$

$$G(j) \{\omega_\nu(j) V_\nu(j , \xi)\}^{-1}$$

is either non-decreasing for $0 \leqq j < \infty$ or it is eventually non-increasing. In the latter case we are done. Thus it is sufficient to show that the case $G(j) \{\omega_\nu(j) V_\nu(j , \xi)\}^{-1}$ non-decreasing for every $\xi > 1$ is impossible. We must divide our argument into three cases.

i. $0 < \nu < 1/2$. Here, see (17) of § (2),

$$\lim_{\xi \to 1+} V_\nu(j , \xi) (\xi - 1)^{1/2 - \nu} = c_\nu \neq 0 \qquad (j = 0, 1, \ldots) .$$

If $G(j) \{\omega_\nu(j) V_\nu(j , \xi)\}^{-1}$ is non-decreasing as a function of j for every $\xi > 1$ then so is

$$G(j) \{\omega_\nu(j)\}^{-1} = \lim_{\xi \to 1+} c_\nu (\xi - 1)^{\nu - 1/2} G(j) \{\omega_\nu(j) V_\nu(j , \xi)\}^{-1} .$$

But $G(j) \{\omega_\nu(j)\}^{-1}$ non-decreasing contradicts our assumption that $G(j) \in l^1_\nu$.

ii. $\nu = 1/2$. Here by formula (17′) of § 2 we see that

$$\lim_{\xi \to 1+} V_\nu(j , \xi) \left\{\log \frac{1}{\xi - 1}\right\}^{-1} = c \neq 0 \qquad (j = 0, 1, \ldots) .$$

If $G(j) \{\omega_\nu(j) V_\nu(j , \xi)\}^{-1}$ is non-decreasing for every $\xi > 1$ then so is

$$G(j)\{\omega_\nu(j)\}^{-1} = \lim_{\xi\to 1+} c \log\left(\frac{1}{\xi-1}\right) G(j)\{\omega_\nu(j) V_\nu(j,\xi)\}^{-1},$$

etc.

iii. $\nu > 1/2$. Here, see (17″) of § 2,

$$\lim_{\xi\to 1+} V_\nu(j,\xi) = c_\nu \frac{\Gamma(j+1)}{\Gamma(j+2\nu)} \qquad (j = 0, 1, \ldots).$$

If $G(j)\{\omega_\nu(j) V_\nu(j,\xi)\}^{-1}$ is non-decreasing for every $\xi > 1$ then so is

$$G(j)\,\omega_\nu(j)^{-1}\,\frac{\Gamma(j+2\nu)}{\Gamma(j+1)} = \lim_{\xi\to 1+} c_\nu G(j)\{\omega_\nu(j) V_\nu(j,\xi)\}^{-1},$$

etc.

6. Variation diminishing kernels.

We can now establish the converse of Theorem 4b.

Definition 6a. A matrix[4] $[a(n,k)]$, $n_1 < n < n_2$, $k_1 < k < k_2$ is said to be variation diminishing if

$$\psi(n) = \sum_{k_1<k<k_2} a(n,k)\,\varphi(k) \qquad n_1<n<n_2$$

implies $\mathrm{V}[\psi] \leq \mathrm{V}[\varphi]$ for every function $\varphi(k)$, $k_1 < k < k_2$, which is zero except a finite number of values of k.

Lemma 6c. If $G(n)$ is a variation diminishing kernel in l_ν^1 and if

$$a(n,k) = \sum_{j=0}^{\infty} C_\nu(n,j,k)\,\omega_\nu(k)\,\omega_\nu(j)\,G(j) \tag{1}$$

then $[a(n,k)]$ $(0 \leq n < \infty,\ 0 \leq k < \infty)$ is a variation diminishing matrix.

Note that the sum on the right side of (1) is finite. By assumption

$$\mathrm{V}[G \# \varphi] \leq \mathrm{V}[\varphi]$$

for any $\varphi(k) \in l_\nu^\infty$ and therefore a fortiori for any $\varphi(k)$ defined for $k = 0, 1, \ldots$ and zero except for finitely many values of k. By (2) of § 2 we see that

4. Here n_1 and k_1 may be finite or $-\infty$ and n_2 and k_2 may be finite or $+\infty$.

$$G \# \varphi \cdot (n) = \sum_{k=0}^{\infty} a(n, k) \varphi(k)$$

and our result follows.

Let us define $E(r, s)$, $0 \leqq r, s < \infty$ by the formulas

$$E(2r, 2s) = \frac{(\nu)_{s-r}(\nu)_{r+s}(1)_{2s}}{(1)_{s-r}(1)_{r+s}(2\nu)_{2s}},$$

$$E(2r+1, 2s+1) = \frac{(\nu)_{s-r}(\nu)_{r+s+1}(1)_{2s+1}}{(1)_{s-r}(1)_{r+s+1}(2\nu)_{2s+1}},$$

$$E(2r, 2s+1) = E(2r+1, 2s) = 0,$$

if $s \geqq r$. If $s < r$ then $E(r, s) = 0$.

Theorem 6d. *Let $-\infty < r < \infty$, and $s \geqq 0$; we have*

$$\lim_{n \to \infty} C_\nu(n, n+r, s)\, \omega_\nu(n) = E(|r|, s).$$

Moreover for each r there is a constant $A(\nu, r)$ such that

$$C_\nu(n, n+r, s)\, \omega_\nu(n) \leqq A(\nu, r)(s+1)^\mu E(|r|, s)$$

for $0 \leqq n$, $|r| \leqq s$. Here $\mu = \max(2\nu - 1, 1)$.

Note that if $s < |r|$ then $C_\nu(n, n+r, s)$ vanishes for all n. Let us first suppose that $r \geqq 0$. We have

$$\begin{aligned} & C_\nu(n, n+2r, 2s)\, \omega_\nu(n) \\ &= \left\{\frac{(\nu)_{n+r-s}(1)_{n+2r}(2\nu)_{n+r+s}}{(1)_{n+r-s}(2\nu)_{n+2r}(\nu)_{n+r+s}}\right\} \times \left\{\frac{n+\nu}{n+r+s+\nu}\right\} \\ &\times \left\{\frac{(\nu)_{s-r}(\nu)_{r+s}(1)_{2s}}{(1)_{s-r}(1)_{r+s}(2\nu)_{2s}}\right\} \times \left\{\frac{\sqrt{\pi}\, 2^{1-2\nu}\Gamma(2\nu)}{\Gamma(\nu)\Gamma(\nu+1/2)}\right\} \\ &= F_1 F_2 F_3 F_4 . \end{aligned}$$

By the Legendre duplication formula $F_4 = 1$. Also $0 \leqq F_2 \leqq 1$ and $\lim_{n\to\infty} F_2 = 1$. It is easily verified that $\lim_{n\to\infty} F_1 = 1$. Using the elementary inequality

$$(\alpha)_m / (\beta)_m \leqq A(\alpha, \beta)(m+1)^{\alpha-\beta} \qquad m \geqq 0$$

we find that $0 \leqq F_1$ and that

$$F_1 \leqq \mathbf{A}(\nu)(n+r-s+1)^{\nu-1}(n+2r+1)^{1-2\nu}(n+r+s+1)^\nu .$$

If $\nu \geqq 1$ this implies that

$$F_1 \leqq A(\nu)(n+r+s+1)^{\nu-1}(n+2r+1)^{1-2\nu}(n+r+s+1)^{\nu},$$
$$F_1 \leqq A_1(\nu, r)(s+1)^{2\nu-1} \qquad \nu \geqq 1.$$

For $0<\nu<1$ we must divide the argument into cases. If $r \leqq s \leqq {}^1/_2 n$ then

$$F_1 \leqq A(\nu)(\tfrac{1}{2}n+r+1)^{\nu-1}(n+2r+1)^{1-2\nu}(\tfrac{3}{2}n+r+1)^{\nu}$$
$$\leqq A_1(\nu, r).$$

If $r \leqq s$, ${}^1/_2 n < s$ then

$$F_1 \leqq A(\nu)(n+2r+1)^{1-2\nu}(n+r+s+1)^{\nu}$$
$$\leqq A(\nu)(n+2r+1)^{1-\nu}(n+2r+1)^{-\nu}(n+r+s+1)^{\nu}$$
$$\leqq A(\nu)(2s+2r+1)^{1-\nu}(n+2r+1)^{-\nu}(n+r+s+1)^{\nu}$$
$$\leqq A_1(\nu, r)(s+1)^{1-\nu}(s+1)^{\nu} = A_1(\nu, r)(s+1).$$

Let us now turn to the case $r<0$. Let $r=-r_1$ where $r_1>0$. If $n_1=n-r_1$ then since C is symmetric in its arguments

$$C_\nu(n, n+r, s)\,\omega_\nu(n)$$
$$= C_\nu(n_1, n_1+r_1, s)\,\omega_\nu(n_1)\,[\omega_\nu(n_1+r_1)/\omega_\nu(n_1)].$$

Since $C_\nu(n, n+r, s)$ is 0 if $n+r<0$, that is if $n_1<0$, and since

$$0 \leqq \omega(n_1+r)/\omega(n_1) \leqq A_1(\nu, r) \qquad n_1 \geqq 0,$$

the case $r<0$ follows from the case $r \geqq 0$, etc.

For $G(j) \in l^1$ we define

$$g(r) = \sum_{j=0}^{\infty} E(|r|, j)\,\omega_\nu(j)\,G(j) \qquad -\infty<r<\infty. \tag{2}$$

Since $E(|r|, j)$ is for fixed r, uniformly bounded in j the infinite sum on the right hand side of (2) is absolutely convergent. Note that $g(r)$ is even.

Theorem 6e. *If $G(j)$ is a variation diminishing kernel in l^1_ν then the matrix $[g(r-s)]$, $-\infty<r, s<\infty$ is variation diminishing. Moreover*

$$\sum_{r=-\infty}^{\infty} g(r)\,e^{ir\vartheta} = G^{\wedge}(\cos\vartheta). \tag{3}$$

We assert first that if $a(n, k)$ is defined by (1) then

$$\lim_{n\to\infty} a(n+r, n+s) = g(r-s). \tag{4}$$

We have

$$a(n+r\,,\,n+s)=\sum_{j=0}^{\infty}C_\nu(n+r\,,\,n+s\,,\,j)\,\omega_\nu(n+s)\,\omega_\nu(j)\,G(j)$$

$$=\sum_{j=0}^{\infty}C_\nu(m+r-s\,,\,m\,,\,j)\,\omega_\nu(m)\,\omega_\nu(j)\,G(j)$$

where $m=n+s$.

Using Theorem 5c, Theorem 6d, and the series analogue of the Lebesgue dominated convergence theorem we obtain (4). The first conclusion of our theorem follows in an evident fashion from (4) and Lemma 6c.

To prove (3) we begin by noting that

$$\sum_{r=-\infty}^{\infty}E(|r|\,,\,j)\,e^{ir\vartheta}=W_\nu(j\,,\,\cos\vartheta)\,. \tag{5}$$

This is, apart from a slight change of notation, formula (17) of [6; vol. II, p. 175]. (Note that $E(|r|,j)=0$ if $|r|>j$). Thus formally we have

$$\begin{aligned}\sum_{r=-\infty}^{\infty}g(r)\,e^{ir\vartheta}&=\sum_{r=-\infty}^{\infty}e^{ir\vartheta}\sum_{j=0}^{\infty}E(|r|\,,\,j)\,\omega_\nu(j)\,G(j)\\&=\sum_{j=0}^{\infty}G(j)\,\omega_\nu(j)\sum_{r=-\infty}^{\infty}e^{ir\vartheta}E(|r|\,,\,j)\\&=\sum_{j=0}^{\infty}G(j)\,\omega_\nu(j)\,W_\nu(j\,,\,\cos\vartheta)\\&=G^{\wedge}(\cos\vartheta)\,.\end{aligned} \tag{6}$$

Note that the $E(|r|\,,\,j)$'s are non-negative and setting $\vartheta=0$ in (5)

$$\sum_{r=-\infty}^{\infty}E(|r|\,,\,j)=1\,.$$

Thus the iterated sum (6) is absolutely convergent since $G\in l^1_\nu$. This justifies our change of orders of summation.

Theorem 6f. *If $G(j)$ is a variation diminishing kernel in l^1_ν then*

$$G^{\wedge}(x)=de^{cx}\,\frac{\prod_{k=1}^{\infty}(1+a_k\,x)}{\prod_{k=1}^{\infty}(1+b_k\,x)}\qquad -1\leqq x\leqq 1$$

where

$$c \geqq 0, \quad 1 \geqq a_k \geqq 0, \quad 1 > b_k \geqq 0, \quad \sum_{k=1}^{\infty} a_k + b_k < \infty .$$

By a theorem of Schoenberg and Motzkin, see [9, Chapter V], if $[g(r-s)]$, $-\infty < r, s < \infty$ is variation diminishing and if $R[g]$ is the rank of the matrix $[g(r-s)]$ then for $k < R[g]$ all determinants

$$\det [g(r_i - s_j)]_{i,j=1}^{k} \tag{7}$$

where $r_1 < r_2 < \dots < r_k$, $s_1 < s_2 < \dots < s_k$ are of the same sign. By Lemma 6a we may assume that $G(j) \geqq 0$ for $0 \leqq j < \infty$. It follows from (2) that $g(r) \geqq 0$ for $-\infty < r < \infty$. By (3) with $\vartheta = 0$ we see that

$$\sum_{r=-\infty}^{\infty} g(r) = G^{\wedge}(1) = \|G\|_1 < \infty .$$

It follows that $\lim_{|r|\to\infty} g(r) = 0$. Unless $G \equiv 0$ we will have $g(m) > 0$ for some m. Let $r_i = m + iN$, $i = 1, \dots, k$ and $s_j = jN$, $j = 1, \dots, k$. Here N is a (large) integer which will be specified presently. The determinant (7) becomes in this case

$$\det [g(m + \{i-j\}N)]_{i,j=1}^{k} .$$

The diagonal elements of this determinant are all $g(m)$. By choosing N sufficiently large the non-diagonal elements can be made arbitrarily small. Thus for a suitable choice of N the determinant is positive. By the Schoenberg-Motzkin theorem it follows that all the determinants (7) for every k are non-negative — that is the matrix $[g(r-s)]$, $-\infty < r, s < \infty$ is "totally positive".

By Edrei's theorem [5] it follows that (taking into account the evenness of $g(r)$)

$$\sum_{-\infty}^{\infty} g(r) z^r = d_1 e^{\varepsilon(z+z^{-1})} \frac{\prod_{k=1}^{\infty} [1 + \alpha_k (z + z^{-1}) + \alpha_k^2]}{\prod_{k=1}^{\infty} [1 + \beta_k (z + z^{-1}) + \beta_k^2]} \tag{9}$$

5. Edrei gives his result in the form used here not in the form quoted in § 1. The (well known) arguments immediately preceding bridge this discrepancy.

where $\varepsilon \geqq 0$, $0 \leqq \alpha_k$, $0 \leqq \beta_k$, $\beta_k \neq 1$ and $\Sigma_k \alpha_k + \beta_k < \infty$. The Laurent series (9) converges in some annulus [6] $\rho_1 < |z| < \rho_2$ where $\rho_1 < 1 < \rho_2$. Setting $z = e^{i\vartheta}$, $\alpha_k / (1 + \alpha_k^2) = a_k$, $\beta_k / (1 + \beta_k^2) = b_k$, etc., we find that

$$G^{\wedge}(\cos\vartheta) = \sum_{-\infty}^{\infty} g(r)\, e^{ir\vartheta} = d\, e^{c\cos\vartheta} \frac{\prod_{k=1}^{\infty} (1 + a_k \cos\vartheta)}{\prod_{k=1}^{\infty} (1 - b_k \cos\vartheta)}$$

where $c \geqq 0$, $1 \geqq a_k \geqq 0$, $1 > b_k \geqq 0$, $\Sigma_k a_k + b_k < \infty$. Putting $\cos\vartheta = x$ our theorem is proved.

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6. The annulus contains $|z| = 1$ in its interior because

$$\sum_{-\infty}^{\infty} g(r) < \infty .$$

The restriction $\beta_k \neq 1$ is so that there will be no pole on $|z| = 1$.

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ON ANALYTIC ITERATION

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1. Definitions.

Let Ω be the set of all analytic functions $F(z)$ which admit an expansion of the type:

$$F(z) = z + f_2 z^2 + f_3 z^3 + \dots,$$

convergent for:

$$|z| < \rho \quad , \quad \rho > 0 .$$

Let S be a set of complex numbers such that $a \in S$ and $b \in S$ implies $(a-b) \in S$, with $1 \in S$.

The function $F(z)$ will be said to possess iterates in S if there exists a function $F(s, z)$, called the s-iterates of $F(z)$, defined for $s \in S$, satisfying the following four conditions:

(1) $F(1, z) = F(z)$,

(2) $F(s, z) \in \Omega$, $(s \in S)$,

(3) $F[s, F(s', z)] = F[(s+s'), z]$, $(s, s' \in S)$,

(4) $F(s, z) = \sum_{k=1}^{k=\infty} f_k(s) z^k$ (for $s \in S$, $|z| < \rho(s)$, $\rho(s) > 0$),

where $f_k(s)$ are polynomials in s.

If the set S is the set of all integers, $F(s, z)$ is said to be the integer iterate of $F(z)$.

If the set S is the set of all real numbers, $F(s, z)$ is said to be the complete real iterate of $F(z)$.

If the set S is the set of all complex numbers, $F(s, z)$ is said to be the complete complex iterate of $F(z)$.

If $F(s, z)$ is a complete complex iterate of $F(z)$ and is analytic in s, it is said to be the analytic iterate of $F(z)$.

2. The Main Theorem.

The purpose of this paper is to prove that:

If the function $F(z) \in \Omega$ admits a complete real iterate then a function $F(s, z)$ exists which is the complete complex iterate of $F(z)$. This function is analytic in s and is therefore the analytic iterate of $F(z)$. If $F(z)$ does not have an analytic iterate, then it can only have iterates $F(s, z)$ in a real set S of one-dimensional measure zero and in a complex set S of two-dimensional measure zero.

This theorem partially fills the gap in our knowledge about the analiticity (in s) of the s-iterates of analytic functions. Indeed, it is well known [4], [6] that functions of the type $F(z) = f_1 z + f_2 z^2 + \dots$ with $|f_1| \neq 1$ always have complete complex and analytic iterates. The case $|f_1| = 1$, but $f_1 \neq 1$ is still largely open [4]. The case $f_1 = 1$ is the one covered by our Theorem.

The theorem shows that the functions with $f_1 = 1$ fall into two complementary classes: those having a complete complex and analytic iterate and those who have iterates only in a set S of one- or two-dimensional measure zero. The two classes are not void. The function

$$F(s, z) = \frac{z}{1 - sz}$$

is a classical example of an analytic iterate. The function $e^z - 1$ was shown by I. N. Baker [1] to have no real non-integer iterates. M. Levine [6] showed, using some results of the present paper, that this function and the functions $z + z^2$ and $\dfrac{z}{(1-z)^2}$ have no analytic iterate.

To prove our theorem we need some classical results from the theory of integer iterates of functions in Ω.

3. The integer iterates.

If S is the set of all integers, the condition (4) in our definition of $F(s, z)$ is redundant. More precisely we have the following well known result which we quote in a form convenient for us and prove for completeness' sake:

Let $F(z) \in \Omega$ have the expansion:

$$F(z) = z + f_2 z^2 + f_3 z^3 + \dots \qquad (\text{for } |z| < \rho,\ \rho > 0).$$

Then the s-iterate $F(s, z)$ of $F(z)$, for integer s, which satisfies conditions (1), (2) and (3) is uniquely defined and can be expanded in the power series:

$$F(s, z) = \sum_{k=1}^{k=\infty} f_k(s) z^k \qquad (\text{for } |z| < \rho(s),\ \rho(s) > 0), \tag{5}$$

where the functions $f_k(s)$ are polynomials in s (so that condition (4) is automatically satisfied) of degree $n \leqq k-1$. Furthermore:

$$f_k(0) = \delta_{k,i}\ ;\ f_k(1) = f_k\ ;\ f_1(s) = 1 \tag{6}$$

and the degree of $f_k(s)$ is $n \leqq k-1$.

Proof [3]: Let $m > 0$ be an integer. Expand $[F(z)]^m$ in a power series and put:

$$[F(z)]^m = \sum_{k=1}^{k=\infty} f_{m,k} z^k \qquad (\text{with } f_{m,k} = 0 \text{ for } k < m).$$

Consider now the matrix:

$$\mathbf{F} = \| f_{m,k} \|, \qquad (m = 1, 2, \dots;\ k = 1, 2, \dots).$$

It is readily shown by induction, using condition (3) that, for positive integer s and m, if we write:

$$[f(s, z)]^m = \sum_{k=1}^{k=\infty} f_{m,k}(s) z^k$$

and:

$$\mathbf{F}(s) = \| f_{m,k}(s) \|, \qquad (m = 1, 2, \dots;\ k = 1, 2, \dots)$$

then:

$$\mathbf{F}(s) = (\mathbf{F})^s \qquad (\text{equation between matrices}). \tag{7}$$

Noting that the matrix $\mathbf{F}$ is triangular, with all its diagonal elements $= 1$, and denoting by $\mathbf{J}$ the unit matrix:

$$\mathbf{J} = \| \delta_{m,k} \|, \qquad (m = 1, 2, \dots;\ k = 1, 2, \dots),$$

we have, using the fact that the unit matrix commutes with all matrices:

$$\mathbf{F}(s) = [(\mathbf{F}-\mathbf{J})+\mathbf{J}]^s = \sum_{\sigma=0}^{\sigma=s} \binom{s}{\sigma} (\mathbf{F}-\mathbf{J})^\sigma \qquad (\text{where } (\mathbf{F}-\mathbf{J})^0 = \mathbf{J}).$$

Let $(\mathbf{F}-\mathbf{J})^\sigma_{m,k}$ be the element of the m-th row and the k-th column of the matrix $(\mathbf{F}-\mathbf{J})^\sigma$, then:

$$(\mathbf{F}-\mathbf{J})^\sigma_{m,k} = 0 \quad \text{for} \quad \sigma > k-m$$

because the main diagonal of the matrix $(\mathbf{F}-\mathbf{J})$ is zero.

It follows that $f_k(s)$, which is the element $(1, k)$ of the matrix $\mathbf{F}(s)$, is given by:

$$(8) \qquad f_k(s) = \sum_{\sigma=0}^{\sigma=s} \binom{s}{\sigma} (\mathbf{F}-\mathbf{J})^\sigma_{1,k} = \sum_{\sigma=0}^{\sigma=k-1} (\mathbf{F}-\mathbf{J})^\sigma_{1,k} \binom{s}{\sigma}.$$

Thus $f_k(s)$ is a polynomial in s. The highest degree of s occurs in the term with the highest σ. This degree is thus $(k-1)$ or less (if $(\mathbf{F}-\mathbf{J})^{k-1}_{1,k} = 0$).

Noting that $\binom{s}{\sigma} = 0$ if $\sigma > s$ we now easily verify conditions (6).

4. Non integer s.

We now have to examine our definition of $F(s, z)$ for non integer s. Conditions (1) and (3) are unavoidable in any extension of the definition of iteration. Condition (2) is arbitrary but seems to be a natural requirement without which the problem would be too indefinite.

Condition (4), for non integer s, does not result from conditions (1), (2) and (3). Indeed, let $f_k(s)$ be defined by (8) and suppose conditions (1), (2) and (3) to be satisfied. Let $H(s)$ be a Hammel function defined for $s \in S$. That is let:

$$H(s) \not\equiv s, \; H(1) = 1$$

and

$$H(s+s') = H(s) + H(s'), \qquad (s, s' \in S).$$

Then the function:

$$F^*(s\,,z) = \sum_{k=1}^{k=\infty} f_k[H(s)]\,z^k\,,$$

which, for integer s coincides with $F(s\,,z)$, satisfies conditions (1), (2) and (3) for all $s \in S$, but not condition (4).

Condition (4) is therefore necessary to ensure the unicity of the function $F(s\,,z)$ for those values of s for which it is defined. It is also sufficient to ensure this unicity because a polynomial $f_k(s)$ is uniquely determined if its values are given for all integer s.

Note that condition (4) could be replaced, in the case where the set S is dense, by the requirement that the functions $f_k(s)$ be continuous functions of s.

5. The sequence $\{l_k\}$.

Consider the sequence of numbers $\{l_k\}$ defined by:

$$l_k = f'_{k+1}(0)\,.$$

(Note that we could as well have written l_{k+1} for l_k. Our choice of notation is made to conform with other usage). Here $f'_k(s)$ is the derivative of the polynomial $f_k(s)$ defined by equation (8), so that:

$$l_0 = 0\,;\quad l_k = \sum_{\sigma=1}^{\sigma=k} \frac{(-1)^{\sigma-1}}{\sigma}\,(\mathbf{F}-\mathbf{J})^{\sigma}_{1\,,\,k+1}\,,\qquad (k = 1,\,2,\,\ldots)\,. \tag{9}$$

Equations (9) show that the function $F(z)$ determines uniquely the sequence $\{l_k\}$.

Conversely, the sequence $\{l_k\}$ determines uniquely the sequence $\{1\,,f_2\,,f_3\,,\ldots\}$ of the coefficients of the expansion of the function $F(z) \in \Omega$ which generated the sequence $\{l_k\}$.

Indeed the f_i with the highest subscript which appears in equations (9) is f_{k+1} and this appears only in the term for $s = 1$. In that term it appears with the coefficient 1. Therefore, writing equations (9) successively for $k = 1$, $k = 2, \ldots$, we can solve them successively and determine the numbers l_k. Clearly equations (9) cannot yield the coefficient of z in $F(z)$ but this coefficient is known to be 1 because $F(z) \in \Omega$.

6. The function $L(z)$.

Consider the formal expansion:

$$L(z) = \sum_{k=1}^{k=\infty} f'_k(0)\, z^k = \sum_{k=2}^{k=\infty} l_{k-1} z^k . \tag{10}$$

There are two cases according to whether the radius of convergence ρ of the series on the right is positive or is zero.

We note that if $F(z)$ has an analytic iterate $F(s, z)$ then:

$$L(z) = \left.\frac{\partial F(s, z)}{\partial s}\right|_{s=0} .$$

This results from definitions (4) and (10). Furthermore the function $L(z)$ also satisfies the double functional-differential equation [4]:

$$L[F(s, z)] = \frac{\partial F(s, z)}{ds} = L(z) \cdot \frac{\partial F(s, z)}{\partial z} . \tag{11}$$

Indeed, differentiating equation (3) over s' we find:

$$\frac{\partial F(s', z)}{\partial s'} = F_z[s, F(s', z)] = F_s[(s + s'), z] .$$

Putting $s' = 0$ and noting that, by (6), $F(0, z) = z$, we get the second half of equation (11). Differentiating equation (3) over s we find:

$$F_s[s, F(s', z)] = F_s[(s + s'), z]$$

Putting $s = 0$ and changing s' into s we get the first half of equation (11).

All this holds however only if the function $F(z)$ has an analytic iterate. This is not always the case.

We shall prove two theorems, corresponding to the two possibilities, which together will be seen to be equivalent to our Main Theorem.

7. Two theorems.

Theorem I. *If the radius of convergence of the series (10) is $\rho > 0$ then the series defines a function $L(z)$ and permits to construct a uniquely defined function $F(s, z)$ satisfying conditions (1) to (4) for $|z| < \rho(s)$ with $\rho(s) > 0$ for all finite complex s. This function $F(s, z)$ is then analytic*

in s and in z for all finite complex s and for $|z| < \rho(s)$. It is the complete complex iterate of $F(z)$ and is also the analytic iterate of $F(z)$.

Theorem II. If the radius of convergence of the series in (10) is $\rho = 0$, then the radius of convergence $\rho(s)$ of the series $\sum_{k=1}^{k=\infty} f_k(s) s^k$ is $= 0$ for almost all complex s and for almost all real s. The function $F(z)$ then does not admit a complete complex or a complete real iterate.

As there are only two possibilities, the function $F(s, z)$ qua function of s must either be defined for every finite complex s and be analytic in s, or it must exist only for real values of s belonging to a set of one-dimensional measure zero or for complex values of s belonging to a set of two-dimensional measure zero, which is our Main Theorem.

8. Proof of Theorem I.

We assume that the radius of convergence of the series (10) is $\rho > 0$ and propose to prove that then there exists a unique function $F(s, z)$ satisfying conditions (1) to (4) and that this function is analytic in s.

Consider the differential equation

$$\frac{d\zeta}{dz} = \frac{L(\zeta)}{L(z)} = \frac{l_1 \zeta^2 + l_2 \zeta^3 + \dots}{l_1 z^2 + l_2 z^3 + \dots}. \tag{12}$$

This equation has a meaning for $|z|, |\zeta| < \rho$ but it has a singularity for $z = \zeta = 0$ so that Cauchy's existence theorem is not directly applicable to its solution in the neighborhood of $z = 0$. Let the first l_k which is not $= 0$ be l_p and put:

$$\zeta = z + z^{p+1} \eta.$$

Equation (12) becomes:

$$1 + (p+1) z^p \eta + z^{p+1} \frac{d\eta}{dz} = \frac{L(z + z^{p+1}\eta)}{L(z)},$$

or:

$$\frac{d\eta}{dz} = \frac{L(z + z^{p+1}\eta) - [1 + (p+1) z^p \eta] L(z)}{z^{p+1} L(z)}.$$

A short computation based on Taylor's theorem shows immediately that the right hand side is analytic for $\eta = \eta_0$ for any finite complex η_0 and for sufficiently small z. (Indeed for z such that $|z| < \rho$ and $|z + z^{p+1}\eta_0| < \rho$, where η_0 is any finite complex number). Cauchy's existence theorem (*) is now applicable and shows that the equation in η has a unique solution satisfying the initial conditions:

$$z = 0 \; ; \; \eta = \eta_0 .$$

We shall now choose $\eta_0 = l_p s$, where s is another arbitrary finite complex constant.

Equation (12) has therefore a unique solution of the form:

$$\zeta = \zeta(s, z) = z + f'_{p+1}(0)\, s z^{p+1} + z^{p+2}\,\xi(s, z), \tag{13}$$

where $\xi(s, z)$ is analytic in z for any s and for $|z| < \rho(s)$ for some $\rho(s) > 0$.

We now want to show that $\zeta(s, z) = F(s, z)$ and is analytic in s. The proof of this is quite straightforward and elementary but is somewhat lengthy as we have to show that each of the conditions (1) to (4) is satisfied.

Condition (2) is satisfied as, by (13), $\zeta(s, z) \in \Omega$.

To show that condition (3) is satisfied, consider $\zeta(s + s', z)$ and $\zeta(s', z)$. These functions, qua functions of z, satisfy the equations:

$$\frac{d\zeta(s + s', z)}{dz} = \frac{L[\zeta(s + s', z)]}{L(z)}$$

and

$$\frac{d\zeta(s', z)}{dz} = \frac{L[\zeta(s', z)]}{L(z)},$$

with constant s and s'. Whence by division:

$$\frac{d\zeta(s + s', z)}{d\zeta(s', z)} = \frac{L[\zeta(s + s', z)]}{L[\zeta(s', z)]} .$$

But this is an equation of type (12), its solutions are:

$$\zeta(s + s', z) = \zeta[s'', \zeta(s', z)],$$

* E. Goursat, Cours d'Analyse Mathématique. Tome II, p. 347, Paris 1911.

with constant s''. It remains to prove that $s'' = s$. This is seen by expanding both sides into powers of z by (13) and equating the coefficients of z^{p+1}.

To prove that condition (4) is satisfied, that is that the coefficients of the powers of z are polynomials in s, we proceed by induction. By (13) the proposition is true for the coefficients of z^k with $k \leqq p + 1$.

Put

$$\zeta(s, z) = \sum_{k=1}^{k=\infty} g_k(s) z^k$$

and carry this expansion into equation (3). Equating the coefficients of z^σ on both sides of (3) we find, assuming all previous coefficients to be polynomials :

$$g_\sigma(s + s') = g_\sigma(s) + s' P(s, s'),$$

where $P(s, s')$ is a polynomial in s and s'.

Therefore :

$$g'_\sigma(s) = P(s, 0)$$

and $g_\sigma(s)$ is also a polynomial in s.

The function $\zeta(s, z)$ has thus been shown to be the s-iterate of $\zeta(1, z)$. It remains to be shown that it is analytic in s and that condition (1) is satisfied so that $\zeta(1, z) = F_1(z)$.

For $|z| < \rho(s)$ the function $\zeta(s, z)$ is the sum of an absolutely convergent series of analytic functions of s (actually of the polynomials $z^k g_k(\varphi)$). The function $\zeta(s, z)$ is thus analytic in s and it remains to be shown that it is the analytic iterate of $\zeta(1, z)$.

We shall now show that

$$\left.\frac{\partial \zeta(s, z)}{\partial s}\right|_{s=0} = L(z).$$

Put

$$\left.\frac{\partial \zeta(s, z)}{\partial s}\right|_{s=0} = M(z).$$

By equations (11) we have:

$$\frac{\partial \zeta(s, z)}{\partial s} = M[\zeta(s, z)].$$

By equation (12) we have:

$$L(z) \cdot \frac{\partial \zeta(s, z)}{\partial s} = L[\zeta(s, z)].$$

Differentiating this over s we find:

$$L(z) \cdot \frac{\partial}{\partial z}\left[\frac{\partial \zeta(s, z)}{\partial s}\right] = L'[\zeta(s, z)] \cdot \frac{\partial \zeta(s, z)}{\partial s}.$$

Putting $s=0$ and noting that $\zeta(0, z)=z$ (indeed z is a solution of equation (12) and is therefore the unique solution when $s=0$), we get:

$$L(z) \cdot M'(z) = L'(z) \cdot M(z).$$

Therefore $M(z)=cL(z)$ where c is a constant. Using (13) we see that the coefficient of z^{p+1} in $M(z)$ is $f'_{p+1}(0)=l_p$, which is the coefficient of z^{p+1} in $L(z)$ so that $c=1$ and:

$$\left.\frac{\partial \zeta(s, z)}{\partial s}\right|_{s=0} = L(z).$$

But we have seen that $L(z)$ can originate in this way from one function $F(z)$ only, so that:

$$\zeta(1, z) = F(z).$$

and

$$\zeta(s, z) = F(s, z)$$

is the complete complex and the analytic iterate of $F(z)$.

9. Two lemmas.

To prove Theorem II we need two lemmas.

Lemma 1 (real case): Let A and B be given positive numbers with $B \leqq A$. Let $p(x)$ be any polynomial of degree n in x with $|p(0)| \geqq A^k$ where $k \geqq n+1$. Then there exists an absolute constant c such that the one-dimensional measure m_1, of the set in real x for which $|x| < t$ and $|p(x)| < B^k$ is:

$$m_1 \leqq \frac{ct\,B}{A}.$$

Proof: We may suppose $n \geqq 1$, for if $n = 0$ then $m_1 = 0$.

Choose any $(n+1)$ numbers $x_1, x_2, \dots, x_{n+1}$. Then by Lagrange's interpolation theorem:

$$p(x) = \sum_{i=1}^{i=n+1} \frac{s(x)}{s'(x_i)(x - x_i)} p(x_i),$$

where

$$s(x) = \prod_{j=1}^{j=n+1} (x - x_j)$$

and therefore

$$s'(x_i) = \prod_{j \neq i} (x_i - x_j).$$

We have:

$$|p(0)| \leqq \sum_{i=1}^{i=n+1} \frac{|s(0)|}{|s'(x_i)| \cdot |x_i|} |p(x_i)|. \tag{14}$$

Now choose the $(n+1)$ numbers x_i so that:

$$x_i = \text{real}; \quad \frac{\alpha}{A} < |x_i| < t; \quad x_{i+1} - x_i > \frac{\alpha}{nA}; \quad |p(x_i)| < B^k, \tag{15}$$

where α is a positive number to be chosen presently.

If we cannot find $(n+1)$ numbers satisfying (15), then all numbers x_i such that $|p(x_i)| < B^k$ are confined to, at most, n intervals of length $\frac{\alpha}{nA}$. Thus, in this case:

$$m_1 \leqq n \frac{\alpha}{nA} = \frac{\alpha}{A}. \tag{16}$$

If, on the other hand, the $(n+1)$ numbers x_i can be found to satisfy (15), then:

$$A^k \leqq |p(0)|, \quad |s(0)| < t^{n+1}; \quad \frac{1}{|x_i|} < \frac{A}{\alpha} \quad \text{and} \quad |p(x_i)| < B^k.$$

Moreover we can estimate $|s'(x_i)|$ by noting that it takes its smallest value

when the x_j are the closest to each other, that is when

$$x_{j+1} - x_j = \frac{\alpha}{nA}$$

and then for

$$i = \left[\frac{n+1}{2}\right].$$

Therefore:

$$\frac{1}{|s'(x_i)|} \leq \frac{1}{\left\{\left[\frac{n+1}{2}\right]!\right\}^2 \left(\frac{\alpha}{nA}\right)^n} = \frac{n^n}{\left\{\left[\frac{n+1}{2}\right]!\right\}^2} \cdot \frac{A^n}{\alpha^n}.$$

Carrying all these estimates into (14) we get:

$$A^k < t^{n+1} \cdot \frac{A}{\alpha} \cdot B^k \cdot \frac{n^n}{\left[\left\{\frac{n+1}{2}\right]!\right\}^2} \cdot \frac{A^n}{\alpha^n}.$$

Let c_1 be the upper bound of $2\left[\frac{n^n}{\left\{\left[\frac{n+1}{2}\right]!\right\}^2}\right]^{1/(n+1)}$ then the above inequality yields:

$$\alpha < c_1 tA \left(\frac{B}{A}\right)^{k/(n+1)} = c_1 tB \left(\frac{B}{A}\right)^{(k-n-1)/(n+1)} \leqq c_1 tB,$$

because $B \leqq A$ and $k \geqq n+1$.

Choosing $\alpha = c_1 tB$ we arrive at a contradiction so that there are no $(n+1)$ numbers x_i satisfying (15) for this value of α. Therefore, from (16):

$$m_1 \leqq \frac{c_1 tB}{A},$$

which proves Lemma 1.

Lemma 2 (complex case): Let A and B be given positive numbers with $B \leqq A$. Let $p(x)$ be a polynomial of degree n in x with $|p(0)| \geqq A^k$ where $k \geqq n+1$. Then for all complex x there exists an absolute constant c' such that the two-dimensional measure m_2 of the set in complex x for which $|x| < t$ and $|p(x)| < B^k$ is:

$$m_2 \leqq \frac{c' t^2 B}{A}.$$

Proof: As in the proof of Lemma 1 we choose $(n+1)$ numbers x_i but now we demand that these numbers satisfy the conditions:

$$(17) \qquad \sqrt{\frac{\alpha t}{A}} < |x_i| < t; \quad |x_{i+1}| - |x_i| > \frac{\alpha}{nA} \ ^{(*)}; \quad |p(x_i)| < B^k,$$

where α is a positive number to be determined presently. If we cannot find $(n+1)$ such numbers, then all the numbers x for which $|p(x)| < B^k$ are concentrated, at most, in n rings of width $\frac{\alpha}{nA}$ and outer radius $\leqq t$, so that, in this case:

$$(18) \qquad m_2 \leqq n \cdot 2\pi \frac{\alpha}{nA} t = 2\pi \frac{\alpha t}{A}.$$

Proceeding as in the proof of Lemma 1, we find that if $(n+1)$ numbers x_i satisfying (17) exist then:

$$A^k < t^{n+1} \cdot \sqrt{\frac{A}{\alpha t}} \cdot B^k \cdot \frac{n^n}{\left\{\left[\frac{n+1}{2}\right]!\right\}^2} \cdot \frac{A^n}{\alpha^n}.$$

Let c_2 be the upper bound of $\left[\frac{n^n \cdot 2^{n+1}}{\left\{\left[\frac{n+1}{2}\right]!\right\}^2}\right]^{1/(n+1/2)}$, then the above inequality yields:

$$\alpha < c_2 tA \left(\frac{B}{A}\right)^{k/(n+1/2)} = c_2 tB \left(\frac{B}{A}\right)^{(k-n-1/2)/(n+1/2)} \leqq c_2 tB,$$

as before.

Choosing $\alpha = c_2 tB$ we arrive at a contradiction so that there are no $(n+1)$ numbers x_i satisfying (17) for this value of α. Therefore, from (18):

$$m_2 \leqq \frac{2\pi c_2 t^2 B}{A},$$

which proves Lemma 2.

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* We could replace this inequality by $|x_{i+1} - x_i| > \alpha/nA$, then by slightly longer computation we would obtain $m_2 < c'' tB/A$.

10. Proof of Theorem II.

We consider the sequence of polynomials

$$p_k(x) = \frac{1}{x} f_k(x),$$

Let n_k be the degree in x of $p_k(x)$. If $f_2 \neq 0$ then, by (6), $n_k \leqq k-2$. If $f_2 = 0$ the degree n_k is still smaller. In all cases $k > n+1$ and Lemmas 1 and 2 are applicable. As $f_k(0) = 0$ (for $k > 1$), we see that:

$$p_k(0) = f'_k(0) \qquad (\text{for } k \geqq 2).$$

Our assumption is that the series $\sum_{k=2}^{k=\infty} f'_k(0) z^k$ diverges for all $z \neq 0$, that is that, for any given $A > 0$ we have:

$$|p_k(0)| > A^k,$$

for infinitely many k.

Let B be any given positive number. Choose an increasing sequence of positive numbers A_q tending to infinity with q and such that $B \leqq A_1 < A_2 < \dots$. It results from our assumption that, given any q, an integer k_q can be found such that:

$$|p_{k_1}(0) > A_q^{k_q}.$$

We have to prove that, given B, the one- (or two-) dimensional measure m_1 (or m_2) of the set in real (or complex) x for which:

$$\overline{\lim_{q \to \infty}} |p_{k_q}(x)|^{1/k_q} < B, \tag{19}$$

is zero.

It suffices to show this for $|x| < t$. Let $S_{t,B}^{(q)}$ (or $S'^{(q)}_{t,B}$) denote that set in real (or complex) x for which $|p_{k_q}(x)| < B^{k_q}$. Then, by our Lemmas:

$$m_1(S_{t,B}^{(q)}) \leqq \frac{cBt}{A_q} \quad ; \quad m_2(S'^{(q)}_{t,B}) \leqq \frac{c'Bt^2}{A_q}.$$

If x satisfies inequality (19) then $x \in S_{t,B}^{(q)}$ (or $x \in S'^{(q)}_{t,B}$) for all but a finite number of q, so that:

$$x \in \bigcup_{l=1}^{l=\infty} \bigcap_{q=l}^{q=\infty} S_{t,B}^{(q)} \qquad \left(\text{or } x \in \bigcup_{l=1}^{l=\infty} \bigcap_{q=l}^{q=\infty} S'^{(q)}_{t,B}\right). \tag{20}$$

But as $A_q \to \infty$ we have $m_1(S^{(q)}_{t,B}) \to 0$ (or $m_2(S'^{(q)}_{t,B}) \to 0)$. Thus the measure of the sets in x satisfying (20) is zero.

11. General remarks.

Lemmas I and II are akin to H. Cartan's theorem [2]:

Let $h(z) = \prod_{j=1}^{n} (z - z_j)$ be a polynomial and $H > 0$ any given positive number. Then $|f(z)| \geqq \left(\frac{H}{e}\right)^n$ everywhere except in a set covered by, at most, n circles of radius $r_1, r_2, \dots, r_n$ such that $\sum_{i=1}^{n} r_i \leqq 2H$.

Our lemmas can be modified in many ways. In particular in Lemma II it is possible to obtain $m_2 \leqq \frac{c''tB}{A}$ instead of our $m_2 \leqq \frac{c't^2B}{A}$. We have prefered giving the weaker result, which is sufficient for our purposes, because the proof of Lemma II then becomes a repetition of that of Lemma I and is shorter. Similarly if we had taken $k = n + 1$ instead of $k \geqq n + 1$ the condition $A > B$ could be discarded. However, we needed the case $k > n + 1$ because when $f_2 = 0$, the polynomials

$$p_k(x) = \frac{1}{x} f_k(x)$$

are of degree $n < k - 1$.

Our main theorem has been surmised for some time. An attempt to prove it, by using a majorating function, which failed, is described by M. Levin [6].

To G. Szekeres [7] is due a detailed study of the structure of iteration which may yield further results connected with the problem of analyticity. This paper also includes an ample bibliography of the subject.

The authors have vainly attempted, together with G. Szekeres, to give an answer to the following question:

If $F(z)$ admits iterates in the set S but has no analytic iterate, can S be dense on the real axis (or in the complex plane)?

This question is still open. Our Main Theorem only shows that the set S is then of one-dimensional (or two-dimensional) measure 0. The only known result in the field is due to I. N. Barker [1] to the effect that, for the particular function $F(z) = e^z - 1$, the set S on the real axis reduces to the set of integers.

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CONTRACTIONS OF FOURIER COEFFICIENTS AND FOURIER INTEGRALS

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§1. We say that a function $g(x)$ is a contraction of a function $f(x)$ if $|g(x)-g(x')| \leq K|f(x)-f(x')|$, where K is a constant. In other words, if we write $\varphi(f(x)) = g(x)$ where φ is a transformation of f, and if φ belongs to the class Lip 1, we can say that φ is a contraction of $f(x)$. For example, $\varphi(x) = |x|$ satisfies the condition, but this is not analytic.

First we shall consider the Fourier series case. Let us suppose that $F(x)$ is integrable in $(-\pi, \pi)$ and periodic with period 2π and its Fourier series is

$$F(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{1.1}$$

Concerning the contraction of Fourier coefficients of $F(x)$, W. Rudin [16] proved that, in order that the transformation of c_n by φ, that is, $\varphi(c_n)$ shall be Fourier coefficients of a function, it is necessary that φ belongs to the class Lip 1 in a neighborhood of zero. But this condition is not sufficient. For example, there exists a series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ which is a Fourier series, but $\sum_{n=-\infty}^{\infty} |c_n| e^{inx}$ is not (cf. J. P. Kahane [13]). Recently Helson-Kahane-Katznelson-Rudin [10] have proved that $\sum_{n=-\infty}^{\infty} \varphi(c_n) e^{inx}$ is a Fourier series for every Fourier series (1.1), if and only if φ is analytic. It might be interesting to investigate what condition is sufficient for the sequence $\varphi(c_n)$ to be Fourier coefficients of a function, when φ belongs to the class Lip 1.

Turning to the Fourier integral case, we can also consider an analogue: Let $F(x)$ be integrable in $(-\infty, \infty)$ and let its Fourier integral be

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$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(t)\,e^{-itx}\,dt\,.$$

Beurling [1] proved that if there exists a function $\gamma(x)$ such that

$$|F(x)|\leqq\gamma(|x|)$$

in $(-\infty, \infty)$, $\gamma(x)\in L(0, \infty)$ and $\gamma(x)$ is a non-increasing function in $(0, \infty)$, then any contraction $g(x)$ of $f(x)$ is a Fourier integral.

One of the aims of this paper is to give an elementary proof of this theorem of Beurling and also to investigate some L^p-analogues, some Fourier series analogues and their dual theorems. In the last few sections, we shall show some applications of the main theorems.

Our method is based on that used by Prof. Boas [4] to give an elementary proof of one of Beurling's theorems.

The author wishes to express his hearty thanks to Prof. R. P. Boas and Prof. G. Sunouchi for their valuable suggestions and encouragement in the preparation of this paper.

§ **2.** We shall state our main theorems in this section. First we consider the Fourier series cases.

Theorem 1. *Let $F(x)\in L^p(-\pi, \pi)$ $(1\leqq p\leqq 2)$, let it be periodic with period 2π and let its Fourier series be*

$$F(x)\sim\sum_{n=-\infty}^{\infty}c_n e^{inx}\,.$$

For a given sequence $\{d_n\}$, if $d_n\to 0$ $(n\to\pm\infty)$ and

$$|d_n-d_m|\leqq K\,|c_n-c_m|$$

for every n and m, and if there exists a function $\gamma(x)$ such that

(i) $|F(x)|^p\leqq\gamma(|x|)$ *in* $(-\pi, \pi)$,

(ii) $\gamma^{2/p}(x)x^2\in L(0, A)$ *and* $\gamma^{2/p}(x)\in L(A, \pi)$ *for any A such that $\pi>A>0$, and*

(iii) $$\int_0^{\pi}x^{-\frac{3p}{2}}\left(\int_0^{x}\gamma^{\frac{2}{p}}(u)\,u^2\,du\right)^{\frac{p}{2}}dx+\int_0^{\pi}x^{-\frac{p}{2}}\left(\int_x^{\pi}\gamma^{\frac{2}{p}}(u)\,du\right)^{\frac{p}{2}}dx<\infty,$$

then the series $\sum_{n=-\infty}^{\infty} d_n e^{inx}$ is a Fourier series of a function which belongs to $L^p(-\pi, \pi)$.

Corollary 1. Theorem 1 is true when the conditions (ii) and (iii) are replaced by

(ii)′ $$\gamma(x) \in L(0, \pi)$$

and

(iii)′ $\gamma(x)$ is non-increasing in $(0, \pi)$.

The following theorem is a dual to Theorem 1:

Theorem 2. Let

$$F(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

and

$$G(x) \sim \sum_{n=-\infty}^{\infty} d_n e^{inx}.$$

Suppose

$$|G(x) - G(x')| \leqq K |F(x) - F(x')|$$

for any $x, x' \in (-\pi, \pi)$, or more generally,

$$\int_{-\pi}^{\pi} |G(x+h) - G(x)|^2 \, dx \leqq K \int_{-\pi}^{\pi} |F(x+h) - F(x)|^2 \, dx$$

for any h, where K is an absolute constant, and suppose that there exists a positive sequence γ_n such that

(i) $|c_n|^p \leqq \gamma_{|n|}$, and

(ii) $$\sum_{n=1}^{\infty} n^{-3p/2} \Big(\sum_{k=1}^{n} k^2 \gamma_k^{2/p} \Big)^{p/2} + \sum_{n=1}^{\infty} n^{-p/2} \Big(\sum_{k=n+1}^{\infty} \gamma_k^{2/p} \Big)^{p/2} < \infty,$$

then

$$\sum_{n=-\infty}^{\infty} |d_n|^p < \infty,$$

where $0 < p \leqq 2$.

Corollary 2. Theorem 2 is true when the conditions (i) and (ii) are replaced by

(i)′ $\sum_{n=1}^{\infty} \gamma_n < \infty$ and

(ii)′ γ_n is non-increasing when $n = 1, 2, 3, \ldots$, where $2/3 < p \leqq 2$.

Remark I. The case $p = 1$ in Corollary 2 is Beurling's theorem ([1], Th. V) and the case $p = 1$ in Theorem 2 is Boas' theorem ([4]).

In these theorems the case $p = 2$ is trivial.

Remark II. When a complex valued fuuction $\varphi(z)$ transforms a Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ of a function belonging to the class $L^p(-\pi, \pi)$ $(p \geqq 1)$ to the Fourier series $\sum_{n=-\infty}^{\infty} \varphi(c_n) e^{inx}$ of a function belonging to the class $L^q(-\pi, \pi)$, we say φ is of type (L^p, L^q) and we write $\varphi \in (L^p, L^q)$. Following Rudin's argument [16], we may see that if $\varphi \in (L^p, L^q)$, where $p \geqq 1$ and $q \geqq 1$, it is necessary that there exist absolute constants $M > 0$ and $\delta > 0$ such that

$$|\varphi(z)| \leqq M|z| \quad \text{for any} \quad |z| \leqq \delta .$$

Remark III. The statement of Theorem 1 for $p > 2$ does not hold. For example, consider a series

$$\sum_{n=2}^{\infty} \frac{e^{icn(\log n)}}{n^{1/2}(\log n)^{\beta}} e^{inx} = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{2.1}$$

where $c \neq 0$ is a real constant and $\beta > 1$. This is convergent uniformly for $0 \leqq x \leqq 2\pi$ (Zygmund [18], vol. 1, p. 199). Take d_n such that

$$d_n = |c_n| = n^{-1/2}(\log n)^{-\beta},$$

and consider

$$\sum_{n=2}^{\infty} d_n e^{inx} = \sum_{n=2}^{\infty} n^{-1/2}(\log n)^{-\beta} e^{inx}. \tag{2.2}$$

By the Hardy-Littlewood theorem (Zygmund [18], vol. 2, p. 129), a necessary and sufficient condition that (2.2) belongs to L^p $(p > 2)$, is that

$\Sigma d_n^p n^{p-2}$ should be finite. But it is clear that $\Sigma d_n^p n^{p-2} = \infty$ for $p > 2$, and hence (2.2) does not belong to L^p for any $p > 2$.

For suitable choice of signs "$\pm$", a trigonometric series

$$\sum_{n=2}^{\infty} \pm \frac{\cos nx}{n^{1/2} (\log n)^2}$$

is a Fourier series of a continuous function and has the same properties as in the above example (cf. Zygmund [18], vol. 2, p. 101—2).

Remark IV. Concerning the case $p = 1$ in Theorem 2, there exists a function $F(x)$ whose Fourier series converges absolutely, while the Fourier series of $|F(x)|$ does not (cf. Kahane [12]).

Remark V. The statement of Corollary 2 for $p = 2/3$ does not hold. For example, consider $F(x) = e^{ix}$ and

$$G(x) = \sum_{n=2}^{\infty} \frac{e^{icn \log n}}{(n \log n)^{3/2}} e^{inx} .$$

Then there exist positive constants K_1, K_2 and K_3 such that

$$K_1 |h| \leqq |F(x+h) - F(x)| \leqq K_2 |h|$$

and

$$|G(x+h) - G(x)| \leqq K_3 |h|$$

for any h (cf. Zygmund [18], vol. 1, p. 243). Hence $G(x)$ is a contraction of $F(x)$ which has the absolutely convergent Fourier series with a required majorant. But

$$\sum |d_n|^{2/3} = \sum_{n=!}^{\infty} 1 / (n \log n) = \infty .$$

(This remark was indicated by Prof. Zygmund).

Let us turn to the Fourier integral case:

Theorem 3. Suppose that $F(x) \in L^p(-\infty, \infty)$, where $1 \leqq p \leqq 2$, and that its Fourier transform is $f(x)$. Further suppose that $g(x)$ is a contraction of $f(x)$, that is,

$$|g(x) - g(x')| \leqq K |f(x) - f(x')| ,$$

satisfying either, for the case $p = 1$,

$$\lim_{x\to\infty} g(x) = 0$$

or, for the case $1<p<2$,

$$g(x) \in L^{p'}(-\infty, -A) \quad \text{and} \quad \in L^{p'}(A, \infty)$$

where A is a positive finite real number and $p'=p/(p-1)$. If there exists a function $\gamma(x)$ such that

(i) $|F(x)|^p \leqq \gamma(|x|)$,

(ii) $u^2\gamma^{2/p}(u) \in L(0, \delta)$ and $\gamma^{2/p}(u) \in L(\delta, \infty)$ for any $\delta>0$ and (iii)

$$\int_0^\infty x^{-3p/2}\left(\int_0^x u^2\gamma^{2/p}(u)\,du\right)^{p/2} dx + \int_0^\infty x^{-p/2}\left(\int_x^\infty \gamma^{2/p}(u)\,du\right)^{p/2} dx < \infty,$$

then $g(x)$ is a Fourier transform of a function $G(x)$ which belongs to $L^p(-\infty, \infty)$, where $1\leqq p \leqq 2$.

Corollary 3. Theorem 3 is true when the conditions (ii) and (iii) are replaced by

(ii)′ $\gamma(x) \in L(0, \infty)$ and

(iii)′ $\gamma(x)$ is non-increasing in $(0, \infty)$.

§ 3. Proof of Theorem 1.

We may suppose that $1\leqq p<2$. By assumptions (i) and (ii), we have

$$|F(x)|^2 x^2 \leqq \gamma^{2/p}(x)\, x^2 \in L(0, \pi).$$

Hence by the Parseval theorem, we have

$$\sum_{n=-\infty}^{\infty} |c_{n+1}-c_n|^2 = C\int_{-\pi}^{\pi} |F(x)|^2 \sin^2 x/2\,dx \tag{3.1}$$

$$\leqq C\int_0^{\pi} \gamma^{2/p}(x)\sin^2 x/2\,dx < \infty.$$

Since $\{d_n\}$ is a contraction of $\{c_n\}$, we have, by (3.1),

$$\sum_{n=-\infty}^{\infty} |d_{n+1}-d_n|^2 \leqq \sum_{n=-\infty}^{\infty} |c_{n+1}-c_n|^2 < \infty. \tag{3.2}$$

Put $$G_n(x) = \sum_{k=-n}^{n} d_k e^{ikx}, \quad \text{then}$$

$$\sum_{k=-n}^{n} e^{ikx}(d_{k+1} - d_k) = (e^{-ix} - 1) G_n(x) + H_n(x), \quad \text{say.}$$

Since $d_n \to 0$ $(n \to \pm\infty)$, we have by the Parseval theorem,

$$\underset{n\to\infty}{\overset{(2)}{\text{l.i.m.}}}\ H_n(x) = \lim_{n\to\infty} \left(\int_{-\pi}^{\pi} |H_n(x)|^2 dx \right)^{1/2} = 0 .$$

Hence, by (3.2),

$$\underset{n\to\infty}{\overset{(2)}{\text{l.i.m.}}}\ \{G_n(x)(e^{-ix} - 1)\} = \underset{n\to\infty}{\overset{(2)}{\text{l.i.m.}}} \left\{ \sum_{k=-n}^{n} e^{ikx}(d_{k+1} - d_k) \right\},$$

which we can write $\qquad = G(x)(e^{-ix} - 1)$,

where $G(x)$ is defined for almost all x and the Fourier series of $G(x)\ (e^{-ix} - 1)$ is $\sum_{n=-\infty}^{\infty} (d_{n+1} - d_n) e^{inx}$. A simple calculation shows that for any integer α,

$$G(x)(e^{-i\alpha x} - 1) \sim \sum_{n=-\infty}^{\infty} (d_{n+\alpha} - d_n) e^{inx}$$

and

$$(3.3) \quad \frac{1}{\pi} \int_{-\pi}^{\pi} |G(x)(e^{-i\alpha x} - 1)|^2 dx = \frac{4}{\pi} \int_{-\pi}^{\pi} |G(x)|^2 \sin^2(\alpha x)/2\, dx$$

$$= \sum_{n=-\infty}^{\infty} |d_{n+\alpha} - d_n|^2 \leqq \sum_{n=-\infty}^{\infty} |c_{n+\alpha} - c_n|^2$$

$$= \frac{4}{\pi} \int_{-\pi}^{\pi} |F(x)|^2 \sin^2(\alpha x/2)\, dx$$

$$\leqq C \int_{0}^{\pi} \gamma^{2/p}(x) \sin^2(\alpha x/2)\, dx < \infty .$$

The next step is to prove that $G(x) \in L^p(-\pi, \pi)$. Consider (3.3) for the case $\alpha = 1$, then we see that

$$x^2 \mid G(x) \mid^2 \in L(-\pi, \pi), \tag{3.4}$$

and hence

$$\int_{-\pi}^{\pi} |x|^p \mid G(x) \mid^p dx \leqq \left(\int_{-\pi}^{\pi} x^2 \mid G(x) \mid^2 dx\right)^{p/2} \left(\int_{-\pi}^{\pi} dx\right)^{1-\frac{p}{2}} < \infty . \tag{3.5}$$

Also we have

$$\left| \int_{\pm\pi/4}^{\pm\pi} \mid G(x) \mid^p dx \right| \leqq (4/\pi)^p \left| \int_{\pm\pi/4}^{\pm\pi} |x|^p \mid G(x) \mid^p dx \right| < \infty .$$

Thus we have only to show that $\mid G(x) \mid^p \in L(0, \pi/4)$. We have for

$$0 < t \leqq \pi/4 ,$$

$$\int_0^t x^2 \mid G(x) \mid^2 dx \leqq \pi^2 t^2 \int_0^t \mid G(x) \mid^2 \sin^2(x/2t)\, dx .$$

Take an integer α such that $\alpha = [1/t] + 1$. Then

$$|x|/2t \leqq \alpha |x|/2 \leqq \pi/2 \quad \text{for} \quad 0 \leqq x \leqq t < \pi/4 ,$$

since

$$1/t < \alpha \quad \text{and} \quad \alpha |x| \leqq t\{(1/t + 1) + 1\} < \pi .$$

Hence we have

$$\sin^2(x/2t) \leqq \sin^2(\alpha x/2) ,$$

and hence by (3.3)

$$\begin{aligned}
\int_0^t x^2 \mid G(x) \mid^2 dx &\leqq \pi^2 t^2 \int_0^t \mid G(x) \mid^2 \sin^2(x/2t)\, dx \\
&\leqq \pi^2 t^2 \int_{-\pi}^{\pi} \mid G(x) \mid^2 \sin^2(\alpha x/2)\, dx \\
&\leqq C t^2 \int_0^{\pi} \gamma^{2/p}(x) \sin^2(\alpha x/2)\, dx \\
&\leqq C t^2 \left\{\alpha^2 \int_0^t x^2 \gamma^{2/p}(x)\, dx + \int_t^{\pi} \gamma^{2/p}(x)\, dx\right\} .
\end{aligned}$$

Thus we have, for $0<t\leqq\pi/4$,

$$\int_0^t x^2\,|\,G(x)\,|^2\,dx \leqq C\int_0^t x^2\,\gamma^{2/p}(x)\,dx + Ct^2\int_t^{\pi}\gamma^{2/p}(x)\,dx\,. \tag{3.6}$$

By (3.5) we can define

$$\varphi_p(x) = \int_0^x |t|^p\,|\,G(t)\,|^p\,dt\,.$$

Then by Hölder's inequality,

$$|\varphi_p(x)| \leq |x|^{1-p/2}\Big(\int_0^x t^2\,|\,G(t)\,|^2\,dt\Big)^{p/2}$$

$$\leqq C\,|x|^{1-p/2}\Big\{\int_0^x t^2\,\gamma^{2/p}(t)\,dt + x^2\int_x^{\pi}\gamma^{2/p}(t)\,dt\Big\}^{p/2}, \qquad \text{by (3.6)},$$

$$\leqq C\,|x|^{1-p/2}\Big\{\Big(\int_0^x t^2\,\gamma^{2/p}(t)\,dt\Big)^{p/2} + |x|^p\Big(\int_x^{\pi}\gamma^{2/p}(t)\,dt\Big)^{p/2}\Big\}, \tag{3.7}$$

by Jensen's inequality. By the assumptions (iii),

$$C \geqq \int_x^{2x} t^{-3p/2}\Big(\int_0^t \gamma^{2/p}(u)\,u^2\,du\Big)^{p/2}dt\,, \qquad \text{for} \quad \pi/2 \geqq x > 0\,,$$

$$\geqq \Big(\int_0^x \gamma^{2/p}(u)\,u^2\,du\Big)^{p/2}\cdot\int_x^{2x} t^{-3p/2}\,dt$$

$$= (2^{1-3p/2}-1)\,|x|^{1-3p/2}\Big(\int_0^x \gamma^{2/p}(u)\,u^2\,du\Big)^{p/2}.$$

Hence

$$\Big(\int_0^x \gamma^{2/p}(u)\,u^2\,du\Big)^{p/2} \leqq C\,x^{3p/2-1}\,. \tag{3.8}$$

Also

$$C \geq \int_{x/2}^{x} t^{-p/2} \Big(\int_{t}^{\pi} \gamma^{2/p}(u)\, du \Big)^{p/2} dt \geq \Big(\int_{x}^{\pi} \gamma^{2/p}(u)\, du \Big)^{p/2} \int_{x/2}^{x} t^{-p/2}\, dt ,$$

and hence

$$\Big(\int_{x}^{\pi} \gamma^{2/p}(u)\, du \Big)^{p/2} \leq C\, x^{p/2-1} . \tag{3.9}$$

By (3.7), (3.8) and (3.9) we have

$$| \varphi_p(x) | \leq C\, |x|^{1-p/2} \{ |x|^{3p/2-1} + |x|^p\, |x|^{p/2-1} \} \leq C\, |x|^p . \tag{3.10}$$

Now we can show that $| G(x) |^p \in L(0\, ,\, \pi/4)$; that is,

$$\int_{0}^{\pi/4} | G(x) |^p\, dx = \int_{0}^{\pi/4} |x|^{-p}\, d\varphi_p(x)$$

$$= \big[|x|^{-p} \varphi_p(x) \big]_0^{\pi/4} + p \int_{0}^{\pi/4} |x|^{-p-1} \varphi_p(x)\, dx ,$$

where the first part is $O(1)$ because of (3.10) and the second part is, by (3.7), less then

$$C \int_{0}^{\pi/4} |x|^{-p-1}\, |x|^{1-p/2} \Big\{ \Big(\int_{0}^{x} t^2 \gamma^{2/p}(t)\, dt \Big)^{p/2} + |x|^p \Big(\int_{x}^{\pi} \gamma^{2/p}(t)\, dt \Big)^{p/2} \Big\} dx$$

$$\leq C \int_{0}^{\pi} |x|^{-3p/2} \Big(\int_{0}^{x} t^2 \gamma^{2/p}(t)\, dt \Big)^{p/2} dx + C \int_{0}^{\pi} x^{-p/2} \Big(\int_{x}^{\pi} \gamma^{2/p}(t)\, dt \Big)^{p/2} dx ,$$

which is finite because of the assumption (iii). Thus we get

$$| G(x) |^p \in L(0\, ,\, \pi)$$

and similarly

$$| G(x) |^p \in L(-\pi\, ,\, 0)$$

and hence

$$| G(x) |^p \in L(-\pi\, ,\, \pi) .$$

From the preceding argument, we see that $G(x)$ is integrable in

$(-\pi, \pi)$ and so $G(x)$ has a Fourier series which we shall denote by

$$G(x) \sim \sum_{n=-\infty}^{\infty} d_n^* e^{inx}.$$

Then $\sum_{n=-\infty}^{\infty} (d_{n+1}^* - d_n^*) e^{inx}$ and $\sum_{n=-\infty}^{\infty} (d_{n+1} - d_n) e^{inx}$ are the Fourier series of the same function $G(x)(e^{-ix} - 1)$ which belongs to $L^2(-\pi, \pi)$.
By the completness of a trigonometric system, we have

$$d_{n+1} - d_n = d_{n+1}^* - d_n^*,$$

that is,

$$d_{n+1} - d_{n+1}^* = d_n - d_n^*,$$

and hence

$$d_n - d_n^* = d_{n+k} - d_{n+k}^*$$

for any integer k. Let $k \to \infty$ and use d_n, $d_n^* \to 0$ $(n \to +\infty)$, then we have

$$d_n - d_n^* = 0,$$

which completes the proof of Theorem 1.

§ 4. Proof of Corollary 1.

By the assumption (ii)′ and (iii)′, we have $\gamma(x) = O(1/x)$, and hence $\gamma^{2/p}(x) \cdot x^2 = O(x^{2-2/p})$. Since $2 - 2/p > -1$, we see that

$$\gamma^{2/p}(x) \cdot x^2 \in L(0, \pi).$$

Hence it is sufficient to show that the conditions (ii)′ and (iii)′ imply the condition (iii), that is, to show

$$\int_0^\pi x^{-3p/2} \Big(\int_0^x \gamma^{2/p}(u)\, u^2\, du \Big)^{p/2} dx$$

$$+ \int_0^\pi x^{-p/2} \Big(\int_x^\pi \gamma^{2/p}(u)\, du \Big)^{p/2} dx \leq C \int_0^\pi \gamma(x)\, dx < \infty.$$

For this purpose we put

$$A(x) = \left\{ \int_0^x u^2 \gamma^{2/p}(u)\,du \right\}^{p/2},$$

then $A(x)$ is increasing and we have

$$\int_0^\pi x^{-3p/2} A(x)\,dx \leqq \sum_{k=0}^\infty A(\pi/2^k) \int_{\pi/2^{k+1}}^{\pi/2^k} x^{-3p/2}\,dx$$

$$\leqq C \sum_{k=0}^\infty (\pi/2^k)^{1-3p/2} A(\pi/2^k) = C \sum_{k=0}^\infty (\pi/2^k)^{1-3p/2} \left\{ \sum_{j=k}^\infty \int_{\pi/2^{j+1}}^{\pi/2^j} u^2 \gamma^{2/p}(u)\,du \right\}^{p/2}$$

$$\leqq C \sum_{k=0}^\infty (\pi/2^k)^{1-3p/2} \left\{ \sum_{j=k}^\infty \gamma^{2/p}(\pi/2^{j+1}) \cdot (\pi/2^j)^3 \right\}^{p/2}$$

$$\leqq C \sum_{k=0}^\infty (\pi/2^k)^{1-3p/2} \sum_{j=k}^\infty \gamma(\pi/2^{j+1}) \cdot (\pi/2^j)^{3p/2}$$

$$= C \sum_{j=0}^\infty \gamma(\pi/2^{j+1}) \cdot (\pi/2^j)^{3p/2} \sum_{k=0}^j (\pi/2^k)^{1-3p/2}$$

$$\leqq C \sum_{j=0}^\infty \gamma(\pi/2^{j+1}) \cdot (\pi/2^j) \leqq C \sum_{j=0}^\infty \gamma(\pi/2^{j+1}) \cdot (\pi/2^{j+2})$$

$$\leqq C \int_0^\pi \gamma(x)\,dx.$$

Similarly we have

$$\int_0^\pi x^{-p/2} \left(\int_x^\pi \gamma^{2/p}(u)\,du \right)^{p/2} dx$$

$$= \sum_{k=0}^\infty \int_{\pi/2^{k+1}}^{\pi/2^k} x^{-p/2} \left(\int_x^\pi \gamma^{2/p}(u)\,du \right)^{p/2} dx$$

$$\leqq \sum_{k=0}^\infty \int_{\pi/2^{k+1}}^{\pi/2^k} x^{-p/2} \left(\sum_{j=0}^k \int_{\pi/2^{j+1}}^{\pi/2^j} \gamma^{2/p}(u)\,du \right)^{p/2} dx$$

$$\leqq C \sum_{k=0}^\infty \{(\pi/2^k)^{1-p/2} - (\pi/2^{k+1})^{1-p/2}\} \left\{ \sum_{j=0}^k \gamma^{2/p}(\pi/2^{j+1}) \cdot (\pi/2^j) \right\}^{p/2}$$

$$\leq C \sum_{k=0}^{\infty} \{(\pi/2^k)^{1-p/2} - (\pi/2^{k+1})^{1-p/2}\} \sum_{j=0}^{k} \gamma(\pi/2^{j+1}) \cdot (\pi/2^j)^{p/2}$$

$$\leq C \sum_{j=0}^{\infty} \gamma(\pi/2^{j+1}) \cdot (\pi/2^j)^{p/2} \sum_{k=j}^{\infty} \{(\pi/2^k)^{1-p/2} - (\pi/2^{k+1})^{1-p/2}\}$$

$$= C \sum_{j=0}^{\infty} \gamma(\pi/2^{j+1}) \cdot (\pi/2^j) \leq C \int_0^{\pi} \gamma(x)\, dx,$$

which completes the proof of Corollary 1.

§ 5. Proof of Theorem 2.

We may suppose $2 > p > 0$. By the Parseval formula, we have

$$\sum_{k=-\infty}^{\infty} |d_k|^2 \sin^2 kh = 1/\pi \int_{-\pi}^{\pi} |G(x+h) - G(x)|^2\, dx \tag{5.1}$$

$$\leq \frac{1}{\pi} \int_{-\pi}^{\pi} |F(x+h) - F(x)|^2\, dx = \sum_{k=-\infty}^{\infty} |c_k|^2 \sin^2 kh.$$

Let us put

$$\varphi_n = \sum_{k=1}^{n} k^p |d_k|^p,$$

then by Hölder's inequality, we have

$$\varphi_n \leq n^{1-p/2} \left(\sum_{k=1}^{n} k^2 |d_k|^2 \right)^{p/2}. \tag{5.2}$$

Then

$$\sum_{n=1}^{N} |d_n|^2 = \sum_{n=1}^{N} n^{-p} (\varphi_n - \varphi_{n-1}) = \sum_{n=1}^{N-1} \varphi_n / n^{1+p} + \varphi_N / N^p$$

$$\leq \sum_{n=1}^{N-1} n^{-3p/2} \left(\sum_{k=1}^{n} k^2 |d_k|^2 \right)^{p/2} + N^{1-3p/2} \left(\sum_{k=1}^{N} k^2 |d_k|^2 \right)^{p/2} = S_1 + S_2, \text{ say.}$$

Since $|c_k|^2 \leq \gamma_k^{2/p}$, we have

$$\sum_{k=1}^{N} k^2 |d_k|^2 \leqq N^2 \sum_{k=1}^{N} |d_k|^2 \sin^2 \frac{k\pi}{2N} \leqq N^2 \sum_{k=-\infty}^{\infty} |d_k|^2 \sin^2 \frac{k\pi}{2N}$$

$$\leqq N^2 \sum_{k=-\infty}^{\infty} |c_k|^2 \sin^2 \frac{k\pi}{2N}, \quad \text{by (5.1)}$$

$$\leqq 2N^2 \sum_{k=0}^{\infty} \gamma_k^{2/p} \sin^2 \frac{k\pi}{2N}. \tag{5.3}$$

By this inequality, we have

$$S_1 \leqq \sum_{n=1}^{N-1} n^{-3p/2} \left\{ n^2 \sum_{k=0}^{\infty} \gamma_k^{2/p} \sin^2 (k\pi/2n) \right\}^{p/2}$$

$$\leqq \sum_{n=1}^{N-1} n^{-3p/2} \left\{ C\, n^2 \sum_{k=1}^{n} \gamma_k^{2/p} k^2/n^2 + n^2 \sum_{k=n+1}^{\infty} \gamma^{2/p} \right\}^{p/2}$$

$$\leqq C \sum_{n=1}^{N-1} n^{-3p/2} \left(\sum_{k=1}^{n} k^2 \gamma_k^{2/p} \right)^{p/2} + C \sum_{n=1}^{N-1} n^{-p/2} \left(\sum_{k=n+1}^{\infty} \gamma_k^{2/p} \right)^{p/2}$$

$$= O(1).$$

The assumption

$$\sum_{n=1}^{\infty} n^{-3p/2} \left(\sum_{k=1}^{n} k^2 \gamma_k^{2/p} \right)^{p/2} < \infty$$

implies

$$\left(\sum_{k=1}^{N} k^2 \gamma_k^{2/p} \right)^{p/2} = O(N^{3p/2-1}), \tag{5.4}$$

because

$$C \geqq \sum_{n=N}^{2N} n^{-3p/2} \left(\sum_{k=1}^{n} k^2 \gamma_k^{2/p} \right)^{p/2} \geqq \left(\sum_{k=1}^{N} k^2 \gamma_k^{2/p} \right)^{p/2} \cdot C' \cdot N^{1-3p/2}.$$

Similarly the other assumption

$$\sum_{n=1}^{\infty} n^{-p/2} \left(\sum_{k=n+1}^{\infty} \gamma_k^{2/p} \right)^{p/2} < \infty$$

implies

$$\left(\sum_{k=N}^{\infty} \gamma_k^{2/p} \right)^{p/2} = O(N^{p/2-1}). \tag{5.5}$$

By (5.2), (5.3), (5.4) and (5.5), we have

$$\varphi_N \leqq N^{1-p/2}\left(N^2\sum_{k=0}^{\infty}\gamma_k^{2/p}\sin^2(k\pi/2N)\right)^{p/2}$$

$$\leqq C\,N^{1-p/2}\left\{\sum_{k=1}^{N}k^2\gamma_k^{2/p}+N^2\sum_{k=N+1}^{\infty}\gamma_k^{2/p}\right\}^{p/2}$$

$$\leqq C\,N^{1-p/2}\left(\sum_{k=1}^{N}k^2\gamma_k^{2/p}\right)^{p/2}+C\,N^{1+p/2}\left(\sum_{k=N+1}^{\infty}\gamma_k^{2/p}\right)^{p/2}$$

$$=O(N^{1-p/2}N^{3p/2-1})+O(N^{1+\frac{p}{2}}N^{p/2-1})$$

(5.6) $$=O(N^p).$$

By (5.6) we have $S_2=\varphi_n/N=O(1)$, which shows that

$$\sum_{k=1}^{\infty}{}'|d_k|^p<\infty.$$

Similarly we have

$$\sum_{k=-\infty}^{-1}|d_k|^p<\infty,$$

and hence

$$\sum_{k=-\infty}^{\infty}|d_k|<\infty,$$

which completes the proof of Theorem 2.

§ 6. Proof of Corollary 2.

We have to show that

(6.1) $$\sum_{n=1}^{\infty}n^{-3p/2}\left(\sum_{k=1}^{n}k^2\gamma_k^{2/p}\right)^{p/2}+\sum_{n=1}^{\infty}n^{p/2}\left(\sum_{k=n+1}^{\infty}\gamma_k^{2/p}\right)^{p/2}<C\sum_{k=1}^{\infty}\gamma_k,$$

under the assumptions (i)′ and (ii)′, which is a corollary of the following inequalities proved by Konyuskov [14].

(I) If $0<\alpha<1$, $\gamma>\alpha-1$ and $d_n\geqq 0$, and if $\{n^{-r}d_n\}$ is almost decreasing[2] for some $r>0$, then

2. When there exists an absolute constant K such that $a_m\leqq Ka_n$ for any $m\geqq n$, we say that a sequence $\{a_n\}$ is almost decreasing. It is obvious that if $\{a_n\}$ is decreasing, then $\{n^{-r}a_n\}$ is almost decreasing for any $r>0$.

$$\sum_{n=1}^{\infty} n^{\gamma-\alpha} \left(\sum_{k=n}^{\infty} d_k \right)^{\alpha} \leqq C \sum_{n=1}^{\infty} n^{\gamma} d_n^{\alpha}.$$

(II) If $0<\alpha<1$, $s>1$ and $d_n \geqq 0$, and if for some $r>0$, $\{n^{-r} d_n\}$ is almost decreasing, then

$$\sum_{n=1}^{\infty} n^{-s} \left(\sum_{m=1}^{n} d_m \right)^{\alpha} \leqq C \sum_{n=1}^{\infty} n^{-s} (n\, d_n)^{\alpha}.$$

§ 7. Proof of Theorem 3. Let us proceed to prove Theorem 3. We may suppose that $1 \leqq p < 2$. For the case $1<p<2$, we have the Fourier transform f of F in the following way:

$$(7.1) \qquad f(t) = \underset{\omega \to \infty}{\overset{(p')}{\text{l.i.m.}}} \frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\omega} e^{-ixt} F(x)\, dx,$$

where $p' = p/(p-1)$. For each real number α, write

$$F_\alpha(x) = (e^{-i\alpha x} - e^{i\alpha x}) F(x) \quad \text{and} \quad f_\alpha(t) = f(t+\alpha) - f(t-\alpha).$$

Then

$$(7.2) \quad f_\alpha(t) = \underset{\omega \to \infty}{\overset{(p')}{\text{l.i.m.}}} \frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\omega} e^{-ixt} F_\alpha(x) = \lim_{\omega \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\omega} e^{-ixt} F_\alpha(x)\, dx,$$

almost everywhere (cf. Zygmund [18], vol. 2, p. 257).

For the case $p=1$, we have also (7.2).

We have $|F_\alpha(x)| = 2|F(x) \sin x\alpha|$, $|F(x)|^p \leqq \gamma(|x|)$,

$$\gamma^{2/p}(x)\, x^2 \in L(0, \delta) \quad \text{and} \quad \gamma^{2/p}(x) \in L(\delta, \infty)$$

for any $\delta > 0$, and hence

$$(7.3) \qquad \int_{-\infty}^{\infty} |F_\alpha(x)|^2\, dx = K \int_{-\infty}^{\infty} |F(x)|^2 \sin^2 x\alpha\, dx$$

$$\leqq K \int_{0}^{\infty} \gamma^{2/p}(x) \sin^2 x\alpha\, dx$$

$$\leqq K \alpha^2 \int_{0}^{\delta} \gamma^{2/p}(x)\, x^2\, dx + K \int_{\delta}^{\infty} \gamma^{2/p}(x)\, dx < \infty.$$

Therefore we have $F_\alpha(x) \in L^2(-\infty, \infty)$. Applying Plancherel's theorem, we see that there exists a Fourier transform $f_\alpha^*(t) \in L^2(-\infty, \infty)$ such that

$$(7.4) \qquad f_\alpha^*(t) = \underset{\omega\to\infty}{\overset{(2)}{\text{l.i.m.}}} \frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\omega} e^{-itx} F_\alpha(x)\, dx$$

and

$$(7.5) \qquad \int_{-\infty}^{\infty} |f_\alpha^*(t)|^2\, dt = \int_{-\infty}^{\infty} |F_\alpha(x)|^2\, dx .$$

From (7.2) and (7.4), we have $f_\alpha(t) = f_\alpha^*(t)$ almost everywhere, and hence, by (7.5),

$$\int_{-\infty}^{\infty} |f_\alpha(t)|^2\, dt = \int_{-\infty}^{\infty} |F_\alpha(x)|^2\, dx ,$$

that is,

$$(7.6) \qquad \int_{-\infty}^{\infty} |f(t+\alpha) - f(t-\alpha)|^2\, dt = 4 \int_{-\infty}^{\infty} |F(x)|^2 \sin^2 x\alpha\, dx$$

$$\leqq K\left\{\alpha^2 \int_0^{\delta} \gamma^{2/p}(x)\, x^2\, dx + \int_{\delta}^{\infty} \gamma^{2/p}(x)\, dx\right\} < \infty .$$

Since $g(t)$ is a contraction of $f(t)$, using (7.6), we have

$$(7.7) \quad \int_{-\infty}^{\infty} |g(t+\alpha) - g(t-\alpha)|^2\, dt \leqq K \int_{-\infty}^{\infty} |f(t+\alpha) - f(t-\alpha)|^2\, dt$$

$$= K \int_{-\infty}^{\infty} |F(x)|^2 \sin^2 x\alpha\, dx < \infty .$$

Put

$$G_n(x) = (1/\sqrt{2\pi}) \int_{-n}^{n} g(t)\, e^{itx}\, dt ,$$

then

$$(1/\sqrt{2\pi})\int_{-n}^{n} e^{itx}\{g(t+\alpha)-g(t-\alpha)\}\,dt=(e^{-i\alpha x}-e^{i\alpha x})\,G_n(x)+H_n(x)\,,$$

where by Parseval's relation, we have

$$(7.8)\qquad \int_{-\infty}^{\infty}|H_n(x)|^2\,dx\leqq\Sigma\left\{\left|\int_{-n\pm\alpha}^{-n}|g(t)|^2\,dt\right|+\left|\int_{n}^{n\pm\alpha}|g(t)|^2\,dt\right|\right\}.$$

For the case $p=1$, we have $\lim\limits_{x\to\pm\infty} f(x)=0$ because of the Riemann-Lebesgue theorem, and so we have $\lim\limits_{t\to\pm\infty} g(t)=0$ since $g(t)$ is a contraction of $f(x)$. Therefore the right hand side of (7.8) tends to zero as $n\to\infty$ which means

$$\underset{n\to\infty}{\overset{(2)}{\text{l.i.m.}}}\, H_n(x)=0\,.$$

The above fact also holds for the case $1<p<2$, because of the assumption

$$g(t)\in L^{p'}\{(-\infty\,,\,-A)\cup(A\,,\,\infty)\}$$

and of, for example,

$$\int_{n}^{n+\alpha}|g(t)|^2\,dt\leqq K\Big(\int_{n}^{n+\alpha}|g(t)|^{p'}\,dt\Big)^{2/p'}.$$

Hence by (7.7) we have

$$\underset{n\to\infty}{\overset{(2)}{\text{l.i.m.}}}\, G_n(x)\,(e^{-i\alpha x}-e^{i\alpha x})=G(x)\,(e^{-i\alpha x}-e^{i\alpha x})\,,$$

where $G(x)$ is defined for almost all x. For any α,

$$(7.9)\qquad \int_{-\infty}^{\infty}|g(t+\alpha)-g(t-\alpha)|^2\,dt=4\int_{-\infty}^{\infty}|G(x)|^2\sin^2 x\alpha\,dx$$

$$\leqq K\int_{0}^{\infty}\gamma^{2/p}(x)\sin^2 x\alpha\,dx<\infty\,,$$

by (7.7).

The next step is to prove that $G(x) \in L^p(-\infty, \infty)$. We have

$$\int_0^t x^2 |G(x)|^2 dx \leq \pi^2 t^2/4 \int_0^t |G(x)|^2 \sin^2 x/t \, dx \tag{7.10}$$

$$\leq \pi^2 t^2/4 \int_{-\infty}^{\infty} |G(x)|^2 \sin^2 x/t \, dx$$

$$= C t^2 \int_{-\infty}^{\infty} |g(x+1/t) - g(x-1/t)|^2 dx, \qquad \text{by (7.9)},$$

$$\leq C t^2 \int_{-\infty}^{\infty} |f(x+1/t) - f(x-1/t)|^2 dx$$

$$\leq C t^2 \int_{-\infty}^{\infty} |F(x)|^2 \sin^2 x/t \, dx$$

$$\leq C t^2 \int_0^{\infty} \gamma^{2/p}(x) \sin^2 x/t \, dx$$

$$\leq C t^2 \left\{ \int_0^t \gamma^{2/p}(x) x^2/t^2 \, dx + \int_t^{\infty} \gamma^{2/p}(x) \, dx \right\}$$

$$= C \left\{ \int_0^t \gamma^{2/p}(x) x^2 \, dx + t^2 \int_t^{\infty} \gamma^{2/p}(x) \, dx \right\}.$$

Hence we have, for $t > 0$,

$$\int_0^t x^2 |G(x)|^2 dx \leq C \left\{ \int_0^t \gamma^{2/p}(x) x^2 \, dx + t^2 \int_t^{\infty} \gamma^{2/p}(x) \, dx \right\} \tag{7.11}$$

Put

$$\varphi_p(t) = \int_0^t |x|^p |G(x)|^p dx,$$

then, by Hölder's inequality,

$$(7.12) \qquad \varphi_p(t) \leqq t^{1-p/2}\left(\int_0^t |x|^2 \,|\, G(x)|^2\, dx\right)^{p/2}$$

$$\leqq C\, t^{1-p/2}\left\{\int_0^t \gamma^{2/p}(x)\, x^2\, dx + t^2 \int_t^\infty \gamma^{2/p}(x)\, dx\right\}^{p/2}$$

$$(7.13) \qquad \leqq C\, t^{1-p/2}\left(\int_0^t \gamma^{2/p}(x)\, x^2\, dx\right)^{p/2}$$

$$+\; C\, t^{1+p/2}\left(\int_t^\infty \gamma^{2/p}(x)\, dx\right)^{p/2}.$$

By the argument for (3.8) and (3.9), we have

$$\left(\int_0^t \gamma^{2/p}(x)\, x^2\right)^{p/2} = O\,({}^{3p/2-1})$$

and

$$\left(\int_t^\infty \gamma^{2/p}(x)\, dx\right)^{p/2} = O\,(t^{p/2-1})\,,$$

and hence we have, since $p < 2$,

$$(7.14) \qquad \varphi_p(t) = O\,(t^p)\,.$$

Using (7.12) and (7.14) we can show that $G(x) \in L^p(0\,,\,\infty)$. In fact, we have

$$\int_0^\infty |\, G(x)\,|^p\, dx = [x^{-p}\, \varphi_p(x)]_0^\infty + \int_0^\infty x^{-p-1}\, \varphi_p(x)\, dx$$

$$= S_1 + S_2\,,$$

say, where

$$S_2 \leqq C \int_0^\infty x^{-p-1}\, x^{1-p/2}\left\{\int_0^x \gamma^{2/p}(t)\, t^2\, dt + x^2 \int_x^\infty \gamma^{2/p}(t)\, dt\right\}^{p/2}$$

$$\leq C \int_0^\infty x^{-3p/2} \left(\int_0^x u^2 \gamma^{2/p}(u)\, du \right)^{p/2} dx$$

$$+ C \int_0^\infty x^{-p/2} \left(\int_x^\infty \gamma^{2/p}(u)\, du \right)^{p/2} dx < \infty ,$$

by the assumption (iii). By (7.14), we have $S_1 = O(1)$, and hence $G(x) \in L^p(0, \infty)$. Similarly we have $G(x) \in L^p(-\infty, 0)$, and so we get $G(x) \in L^p(-\infty, \infty)$.

Now we have the Fourier transform $g^*(x)$ of $G(x)$ in the following way:

$$g^*(t) = \underset{\omega\to\infty}{\overset{(p')}{\mathrm{l.i.m.}}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x)\, e^{-itx}\, dx , \quad \text{for } 1 < p < 2,$$

or

$$g^*(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x)\, e^{-itx}\, dx , \qquad \text{for } p = 1 .$$

By the same argument which we have used to get (7.2), we have

$$g^*_\alpha(t) = \lim_{\omega\to\infty} \int_{-\omega}^{\omega} G_\alpha(x)\, e^{-itx}\, dx$$

almost everywhere, where

$$g^*_\alpha(t) = g^*(t+\alpha) - g^*(t-\alpha) \quad \text{and} \quad G_\alpha(x) = (e^{-i\alpha x} - e^{i\alpha x})\, G(x) .$$

By (7.9), $G_\alpha(x) \in L^2(-\infty, \infty)$. Hence by the argument which we have applied to $f_\alpha(t)$, we see that

$$g^*_\alpha(t) = \underset{\omega\to\infty}{\overset{(2)}{\mathrm{l.i.m.}}} \frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\omega} G_\alpha(x)\, e^{-itx}\, dx .$$

On the other hand, by the definition of $G(x)$ (cf. (7.7) to (7.9)) and the Plancherel theorem, we have also

$$g_\alpha(t) = \underset{\omega\to\infty}{\overset{(2)}{\mathrm{l.i.m.}}} \frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\omega} G_\alpha(x)\, e^{-itx}\, dx .$$

Therefore $g_\alpha(t)=g^*_\alpha(t)$ almost everywhere, that is,

$$g(t+\alpha)-g(t-\alpha)=g^*(t+\alpha)-g^*(t-\alpha).$$

This means that the function $g(t)-g^*(t)$ is periodic with period 2α. Hence

$$g(t)-g^*(t)=g(t+2k\alpha)-g^*(t+2k\alpha).$$

For the case $p=1$, from the Riemann-Lebesgue theorem, we have

$$\lim_{k\to\infty} g^*(t+2k\alpha)=0 \quad \text{and} \quad \lim_{k\to\infty} g(t+2k\alpha)=0.$$

Therefore

$$g(t)-g^*(t)=0,$$

almost everywhere, which completes the proof of Theorem 3 for the case $p=1$.

For the case $1<p<2$, we have to recall the facts that

$$g(t)\in L^{p'}\{(-\infty, -A)\cup(A, \infty)\} \quad \text{and} \quad g^*(t)\in L^{p'}(-\infty, \infty).$$

Take any finite interval (a, b), then we have

$$\int_a^b |g(t)-g^*(t)|^{p'}\,dt=\int_a^b |g(t+2k\alpha)-g^*(t+2k\alpha)|^{p'}\,dt$$

$$=\int_{a+2k\alpha}^{b+2k\alpha} |g(t)-g^*(t)|^{p'}\,dt$$

which tends to zero as $k\to\infty$. Therefore we have $g(t)-g^*(t)=0$ almost everywhere, which completes the proof of Theorem 3.

§8. Proof of Corollary 3.

Since $\gamma(x)=O(1/x)$, we have $\gamma^{2/p}(x)x^2=O(x^{-2/p+2})$. By the assumption $p\geqq 1$, we see that $\gamma^{2/p}(x)x^2\in L(0, \delta)$. Also we have $\gamma^{2/p}(x)=O(x^{-2/p})$ which belongs to $L(\delta, \infty)$ for $\delta>0$. Hence we have to show that the integral in the assumption (iii) of Theorem 3 is dominated by $C\int_0^\infty \gamma(x)\,dx$.

We have

$$\int_0^\infty x^{-3p/2}\left(\int_0^x \gamma^{2/p}(t)\, t^2\, dt\right)^{p/2} dx + \int_0^\infty x^{-p/2}\left(\int_x^\infty \gamma^{2/p}(t)\, dt\right)^{p/2} dx$$

$$\leqq \left\{\int_0^1 x^{-3p/2}\left(\int_0^x \gamma^{2/p}(t)\, t^2\, dt\right)^{p/2} dx + \int_0^1 x^{-p/2}\left(\int_x^1 \gamma^{2/p}(t)\, dt\right)^{p/2} dx\right\}$$

$$+ \left\{\int_1^\infty x^{-3p/2}\left(\int_0^1 \gamma^{2/p}(t)\, t^2\, dt\right)^{p/2} dx + \int_0^1 x^{-p/2}\left(\int_1^\infty \gamma^{2/p}(t)\, dt\right)^{p/2} dx\right\}$$

$$+ \left\{\int_1^\infty x^{-3p/2}\left(\int_1^x \gamma^{2/p}(t)\, t^2\, dt\right)^{p/2} dx + \int_1^\infty x^{-p/2}\left(\int_x^\infty \gamma^{2/p}(t)\, dt\right)^{p/2} dx\right\}$$

$$= I_1 + I_2 + I_3, \quad \text{say.}$$

By the argument in § 4, we have

$$I_1 \leqq C \int_0^1 \gamma(x)\, dx,$$

and since $\gamma^{2/p}(t)\, t^2 \in L(0, 1)$, $\gamma^{2/p}(t) \in L(1, \infty)$, $3p/2 > 1$ and $p/2 < 1$, we have $I_2 = O(1)$. The argument for I_3 is quite similar to that for I_1. In fact, we have

$$\int_1^\infty x^{-3p/2}\left(\int_1^x \gamma^{2/p}(t)\, t^2\, dt\right)^{p/2} dx$$

$$\leqq \sum_{k=1}^\infty \int_{2^{k-1}}^{2^k} x^{-3p/2}\left\{\sum_{j=1}^k \int_{2^{j-1}}^{2^j} \gamma^{2/p}(t)\, t^2\, dt\right\}^{p/2} dx$$

$$\leqq C \sum_{k=1}^\infty 2^{k(1-3p/2)} \sum_{j=1}^k \gamma(2^{j-1})\, 2^{3pj/2}$$

$$= C \sum_{j=1}^\infty \gamma(2^{j-1})\, 2^j \leqq C \int_1^\infty \gamma(x)\, dx < \infty.$$

Similarly we have

$$\int_1^\infty x^{-p/2}\left(\int_x^\infty \gamma^{2/p}(t)\,dt\right)^{p/2} dx$$

$$\leqq \sum_{j=1}^\infty \int_{2^{j-1}}^{2^j} x^{-p/2}\left(\sum_{k=j}^\infty \int_{2^{k-1}}^{2^k} \gamma^{2/p}(t)\,dt\right)^{p/2} dx$$

$$\leqq C\sum_{j=1}^\infty 2^{j(1-p/2)} \sum_{k=j}^\infty \gamma(2^{k-1})\, 2^{kp/2}$$

$$= C\sum_{k=1}^\infty \gamma(2^{k-1})\, 2^k \leqq C\int_1^\infty \gamma(x)\,dx < \infty .$$

Hence we have $I_2 < \infty$, which completes the proof of Corollary 3.

§**9.** In this section we shall give an application of Theorem 1: Here we shall be concerned with trigonometric series

(9.1) $$a_0/2 + \sum_{n=1}^\infty a_n \cos nx, \quad \text{with } a_n \to 0 \ (n \to \infty).$$

The above series can be written in the form $\sum_{n=-\infty}^\infty c_n e^{inx}$ where

$$c_n = c_{-n} = a_n/2$$

for $n \geqq 0$. So we have

$$|c_n - c_m| = (1/2)\,|a_{|n|} - a_{|m|}|.$$

Therefore Theorem 1 can be stated with the term of $\{a_n\}$, instead of $\{c_n\}$.

Let $\Delta a_n = a_n - a_{n-1}$. It is known that the convergence of $\sum_{n=1}^\infty |\Delta a_n|$ is not sufficient for (9.1) to be a Fourier series (cf. Zygmund [18], vol. I, p. 184). In this connection, the following result is interesting:

Theorem 4. *If $a_n \to 0$ as $n \to \infty$ and if, for a $\gamma > 1$ and $n = 2, 3, \dots$,*

$$|\Delta a_n| \leqq \frac{K}{n(\log n)^\gamma},$$

where K is a constant, then the series (9.1) is a Fourier series.

Proof of Theorem 4. Let us put $a_n^* = 1/(\log n)^\alpha$ for $n = 2, 3, \ldots$, where $\alpha = \gamma - 1 > 0$; then

$$\Delta a_n^* \geq \frac{K_2}{n(\log n)^\gamma}.$$

Hence we have

$$|\Delta a_n| \leqq K_3 |\Delta a_n^*| \qquad \text{for } n = 2, 3, \ldots. \tag{9.2}$$

Since $\{a_n^*\}$ is monotone decreasing, (9.2) implies

$$|a_n - a_m| \leqq K' |a_n^* - a_m^*| \qquad \text{for any } n \text{ and } m, \tag{9.3}$$

that is, the sequence $\{a_n\}$ is a contraction of the series $\{a_n^*\}$. Define

$$F(x) = \sum_{n=2}^{\infty} a_n^* \cos nx. \tag{9.4}$$

Then $F(x)$ is finite except $x = 0$. By a simple calculation, we have (cf. Zygmund [18], vol. I, Th. 2.17, p. 189)

$$F(x) \sim \frac{1}{x(\log 1/x)^\gamma} \qquad \text{as } x \to 0.$$

On the other hand, by the Abel transformation, we have

$$|F(x)| \leqq C/\delta \quad \text{in} \quad 0 < \delta \leqq x \leqq \pi,$$

where C is a constant. Combining these results, we easily see that the series (9.4) is the Fourier series of its sum $F(x)$ (cf. Hardy-Rogosinski [9], Th. 100, p. 91), and $F(x)$ has a non-increasing integrable majorant. Hence by Theorem 1, we get Theorem 4.

§ **10**. We shall give another application of Theorem 1:

Theorem 5. *Let the Fourier series of $F(x)$ be $\sum_{n=1}^{\infty} b_n \sin nx$ and suppose that there exists a function $\gamma(x)$ such that $|F(x)| \leqq \gamma(|x|)$ in $(-\pi, \pi)$, $\gamma(x)$ is non-increasing and integrable in $(0, \pi)$. Then $\sum_{n=1}^{\infty} |b_n| \cos nx$ is a Fourier series.*

In particular, we can take $\gamma(x) = K/\{x(\log 1/x)^\gamma\}$, for some $\gamma > 1$.

For the proof, we write

$$\sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where $c_n = -ib_n/2$, $c_{-n} = ib_n/2$ for $n > 0$ and $c_0 = 0$. Then we have

$$\sum_{n=-\infty}^{\infty} |c_n| e^{inx} = \sum_{n=1}^{\infty} (|c_n| e^{inx} + |c_{-n}| e^{-inx}) = \sum_{n=1}^{\infty} |b_n| \cos nx.$$

Now the theorem can be derived from Theorem 1, immediately.

As an application of Theorem 5, we shall derive the Paley-Wiener theorem from it (cf. Hardy [7], Paley-Wiener [15] and Zygmund [17]).

Corollary (Paley-Wiener). *If $F(x)$ is odd, periodic with period 2π and is positive, decreasing and integrable in $(0, \pi)$, then its conjugate function is integrable.*

By the assumption, we see that $F(x)$ is non-increasing in $(0, 2\pi)$. Therefore we have $b_n \geqq 0$ for all $n = 1, 2, 3, \ldots$ (cf. Hardy-Rogosinski [9], p. 25). Hence by Theorem 5, the series $\sum_{n=1}^{\infty} b_n \cos nx$ is a Fourier series. that is, the conjugate function of $F(x)$ is integrable.

§11. Now we shall show some application of Theorem 2.

In this section we suppose that $f(x)$ is defined in $(0, \pi)$ with period π and vanishes at $x = 0$ and $x = \pi$, and that the sine expansion of $f(x)$ converges absolutely, that is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{and} \quad \sum_{n=1}^{\infty} |b_n| < \infty,$$

where

$$b_n = (2/\pi) \int_0^{\pi} f(x) \sin nx \, dx.$$

Under these conditions, the cosine expansion of $f(x)$ is not necessarily absolutely convergent (cf. Izumi-Tsuchikura [11] and Kahane [12]). Izumi-Tsuchikura [11] and Boas [3] obtained several sufficient conditions for the absolute convergence of the cosine expansion of $f(x)$. In connection with

this subject, we shall show the following result; which is a dual of Theorem 5.

Theorem 6. Suppose that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad , \quad \sum_{n=1}^{\infty} |b_n| < \infty \quad \text{and} \quad f(0) = f(\pi) = 0 .$$

If there exists a monotone decreasing sequence γ_n such that

$$|b_n| \leqq \gamma_n \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n < \infty$$

and if $f(x)$ is non negative in $(0, \pi)$, then the cosine expansion of $f(x)$ is absolutely convergent.

For the proof of Theorem 6, we define a function $F(x)$ such that

$$F(x) = f(x) \quad \text{in} \quad (0, \pi) \quad \text{and} \quad F(x) = -f(-x) \quad \text{in} \quad (-\pi, 0)$$

and extend it periodically with period 2π. Then the Fourier series of $F(x)$ is the same as the sine expansion of $f(x)$, and the Fourier series of $|F(x)|$ is the same as the cosine expansion of $f(x)$. So, applying Theorem 2 (Corollary 2) to $F(x)$, we get Theorem 6.

If $f(x)$ is defined by an absolutely convergent sine series, then the improper integral $\int_{\to 0}^{\pi} x^{-1} f(x)\, dx$ always exists, and if, furthermore, the cosine expansion also converges absolutely, then $|x^{-1} f(x)|$ is necessarily integrable (Boas [2]). Combining this and Theorem 6, we have the following corollary:

Corollary. Under the same assumption as in Theorem 6, we have that $|x^{-1} f(x)|$ is integrable.

§ 12. Let us suppose that $f(x)$ is even and periodic with period 2π, and let its Fourier series be

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx .$$

Define a function $F^*(x)$ such that

$$F^*(x) = \frac{1}{2}\cot\frac{x}{2}\int_0^x f(t)\,dt$$

and let its Fourier series be

$$F^*(x) \sim \sum_{n=0}^{\infty} c_n \cos nx .$$

Then we have the following theorem;

Theorem 7. If $f(x)$ has a l^p $(2 \geqq p \geqq 1)$ Fourier series, that is, $\sum_{n=1}^{\infty} |a_n|^p < \infty$, so does $|F^*(x)|$.

The case for $p=1$ is due to Boas and Izumi [5], and the proof of Theorem 7 is based on their lemma: The hypothesis $\sum_{n=1}^{\infty} |a_n|^p < \infty$ implies that

$$c_n = a_n/2n + \sum_{k=n+1}^{\infty} a_k/k$$

(cf. [5], Lemma 2.1). By an inequality ([8], Th. 331, p. 246)

$$\sum_{n=1}^{\infty} (d_n + d_{n+1} + \dots)^p < C_p \sum_{n=1}^{\infty} (n \cdot d_n)^p ,$$

for $d_n \geqq 0$, and $p > 1$, we have, for $p > 1$,

$$\sum_{n=1}^{\infty} |c_n|^p \leqq \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} |a_k/k| \right)^p \leqq C \sum_{k=1}^{\infty} |a_k|^p < \infty .$$

Therefore the sequence $\{|c_n|^p\}$ has a monotone decreasing majorant

$$\gamma_n = \left(\sum_{k=n}^{\infty} |a_k|/k \right)^p \quad \text{with} \quad \sum_{n=1}^{\infty} \gamma_n < \infty .$$

Hence by virtue of Theorem 2 (Corollary 2), we complete the proof of Theorem 7.

Now we shall consider a dual of Theorem 7. Let us suppose that $f(x) \in L^p(-\pi, \pi)$, $(2 \geqq p \geqq 1)$ and

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx .$$

Define

$$F(x) = \int_x^\pi \frac{f(t)}{2\tan t/2}\,dt, \qquad \text{in } (0, \pi),$$
$$= F(-x), \qquad \text{in } (-\pi, 0).$$

$F(x)$ is the adjoint transformation of $f(x)$ with respect to the transformation in Theorem 7. We have, for $p \geqq 1$,

$$\int_0^\pi \left(\int_x^\pi \frac{|f(t)|}{2\tan t/2}\,dt\right)^p dx \leqq K_p \int_0^\pi |f(x)|^p\,dx$$

(cf. [8], Th. 328, p. 244), which means that $F(x)$ has a L^p-monotone majorant. Let

$$F(x) \sim \Sigma A_n \cos nx,$$

then

$$A_n = (1/n)\sum_{k=1}^n a_k - a_n/2n \qquad \text{(cf. Hardy [6])}.$$

Using the above facts and Theorem 1, we have the following theorem which is a dual of Theorem 7.

Theorem 8. *If $f(x) \in L^p(-\pi, \pi)$ $(2 \geqq p \geqq 1)$, then*

$$\Sigma |A_n| \cos nx$$

is a Fourier series of the same class L^p.

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